Members of Random Closed Sets

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Abstract. The members of Martin-Löf random closed sets under a distribution studied by Barmpalias et al. are exactly the members of Martin-Löf random Galton-Watson trees with survival parameter $\frac{2}{3}$. This follows from an effective version of the result that the Barmpalias et al. distribution is the distribution of the infinitely extendible part of a Galton-Watson tree, conditioned on the event that the tree is infinite. To be such a member, a sufficient condition is to have effective Hausdorff dimension strictly greater than $\gamma = \log_2 \frac{3}{2}$, and a necessary condition is to have effective Hausdorff dimension greater than or equal to γ.

Keywords: random closed sets, computability theory.

1 Introduction

Classical probability theory studies intersection probabilities for random sets. A random set will intersect a given deterministic set if the given set is large, in some sense. Here we study a computable analogue: the question of which real numbers are "large" in the sense that they belong to some Martin-Löf random closed set.

Barmpalias et al. [2] introduced algorithmic randomness for closed sets. Subsequently Kjos-Hanssen [5] used algorithmically random Galton-Watson trees to obtain results on infinite subsets of random sets of integers. Here we show that the distributions studied by Barmpalias et al. and by Galton and Watson are actually equivalent, not just classically but in an effective sense.

For $0 < \gamma < 1$, let us say that a real x is a $\textsc{Member}_\gamma$ if x belongs to some Martin-Löf random closed set according to the Galton-Watson distribution with survival parameter $p = 2^{-\gamma}$. We show that for $p = \frac{2}{3}$, this is equivalent to x being a member of a Martin-Löf random closed set according to the distribution considered by Barmpalias et al. In light of this equivalence, we may state that Barmpalias et al. showed that in effect not every $\textsc{Member}_\gamma$ is ML-random, and that Joe Miller and Antonio Montálban showed that every ML-random real is a $\textsc{Member}_\gamma$; the proof of this result is given in the paper of Barmpalias et al. [2]

2 Equivalence of two models

We write $\Omega = 2^{<\omega}$, and 2^ω, for the sets of finite and infinite strings over $2 = \{0,1\}$, respectively. If $\sigma \in \Omega$ is an initial substring (a prefix) of $\tau \in \Omega$ we write $\sigma \preceq \tau$; similarly $\sigma \prec x$ means that the finite string σ is a prefix of the infinite string $x \in 2^\omega$. The length of σ is $|\sigma|$. We use the standard notation $[\sigma] = \{x : \sigma \prec x\}$, and for a set $U \subseteq \Omega$, $[U]^{\preceq} := \bigcup_{\sigma\in U}[\sigma]$. Let $\mathcal{P}$ denote the power set operation. $\{0,1\}$ plays the role of an alphabet, and a tree is a set of strings over $\{0,1\}$ that is closed under prefixes.

Following Kjos-Hanssen [5], for a real number $0 \le \gamma < 1$ (so $\frac{1}{2} < 2^{-\gamma} \le 1$), let $\lambda_{1,\gamma}$ be the distribution with sample space $\mathcal{P}(\Omega)$ such that each string in Ω has probability $2^{-\gamma}$ of belonging to the random set, independently of any other string. Let λ^*_γ be defined analogously, except that now

$$\lambda^*_\gamma(\{S : S \cap \{\sigma 0, \sigma 1\} = J\} = \begin{cases} 1-p & \text{if } J = \{\sigma 0\} \text{ or } J = \{\sigma 1\}, \text{ and} \\ 2p-1 & \text{if } J = \{\sigma 0, \sigma 1\}, \end{cases}$$

independently for distinct σ, for $p = 2^{-\gamma}$.

For $S \subseteq \Omega$, Γ_S, the tree determined by S, is the (possibly empty) set of infinite paths through the part of S that is downward closed under prefixes:

$$\Gamma_S = \{x \in 2^\omega : (\forall \sigma \prec x)\, \sigma \in S\}.$$

The $(1,\gamma)$-induced distribution $\mathbb{P}_{1,\gamma}$ on the set of all closed subsets of 2^ω is defined by

$$\mathbb{P}_{1,\gamma}(E) = \lambda_{1,\gamma}\{S : \Gamma_S \in E\}.$$

Thus, the probability of a property E of a closed subset of 2^ω is the probability according to $\lambda_{1,\gamma}$ that a random subset of Ω determines a tree whose set of infinite paths has property E. Similarly, let

$$\mathbb{P}^*_\gamma(E) = \lambda^*_\gamma\{S : \Gamma_S \in E\}.$$

By *the Galton-Watson (GW) distribution for survival parameter* $2^{-\gamma}$ we will mean the $(1,\gamma)$-induced distribution. It is also known as the distribution of a *percolation limit set* (see Mörters and Peres [10]). S is called $\lambda_{1,\gamma}$*-ML-random* if for each uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ of subsets of Ω with $\lambda_{1,\gamma}(U_n) \le 2^{-n}$, we have $S \notin \bigcap_n U_n$. In this case Γ_S is called ML-random for the GW distribution.

Lemma 1 (Axon [1]). *For* $2^{-\gamma} = \frac{2}{3}$, $\Gamma \subseteq 2^\omega$ *is* $\mathbb{P}^*_\gamma$*-ML-random if and only if* Γ *is a Martin-Löf random closed set under the distribution studied by Barmpalias et al.*

Thinking of S as a random variable, define further random variables

$$G_n = \{\sigma : |\sigma| = n \;\&\; (\forall \tau \preceq \sigma)\; \tau \in S\}$$

and $G = \bigcup_{n=0}^\infty G_n$. We refer to a value of G as a *GW-tree* when G is considered a value of the random variable under the $\mathbb{P}_{1,\gamma}$ distribution. (A *BBCDW-tree* is a particular value of the random variable analogous to G, for the distribution $\mathbb{P}^*_\gamma$.) We have $G \subseteq S$ and $\Gamma_G = \Gamma_S$; however note that even G may have "dead ends".

For a GW-tree G, let

$$[G] = \{\sigma \in G : G \cap [\sigma] \text{ is infinite}\}.$$

Thus $[G] \subseteq G \subseteq S$, and values of $[G]$ are in one-to-one correspondence with values of Γ_S.

Let e be the extinction probability of a GW-tree with parameter $p = 2^{-\gamma}$,

$$e = \mathbb{P}_{1,\gamma}[\Gamma = \varnothing] = \lambda_{1,\gamma}(\{S : \Gamma_S = \varnothing\}).$$

For any number a let $\overline{a} = 1 - a$.

Lemma 2.

$$e = \overline{p}/p.$$

Proof. Notice that we are not assuming $\langle\rangle \in S$. We have $e = \overline{p} + pe^2$, because there are two ways extinction can happen: (1) $\langle\rangle \notin S$, and (2) $\langle\rangle \in S$ but both immediate extension trees go extinct.

Lemma 3.

$$\mathbb{P}_{1,\gamma}\left[[G] \cap \{\langle 0\rangle, \langle 1\rangle\} = J \mid [G] \neq \varnothing\right] = \mathbb{P}^*_\gamma[G_1 = J].$$

Proof. We claim that both are equal to

$$(2p-1)\cdot \mathbf{1}_{J=\{\langle 0\rangle,\langle 1\rangle\}} + \sum_{i=0}^{1}(1-p)\cdot \mathbf{1}_{J=\{\langle i\rangle\}}.$$

By symmetry, and because the probability that $J = \varnothing$ is 0, it suffices to calculate the probability that $J = \{\langle 0\rangle, \langle 1\rangle\}$. This is $\frac{p(1-e)^2}{1-e} = p(1-e) = 2p-1$, using Lemma 2, because first $\langle\rangle$ survives and then both immediate extensions are non-extinct.

Lemma 4. *Define* $\mathfrak{p}_s = \mathbb{P}_{1,\gamma}(\langle j\rangle \in G \mid [G \cap \langle j\rangle] = \varnothing \;\;\&\;\; \langle\rangle \in G)$ *and* $\lambda_f = \lambda_{1,\gamma}(\cdot \mid [G] = \varnothing \;\;\&\;\; \langle\rangle \in G)$. *Then* $\lambda_f(\langle i\rangle \in G_1) = \mathfrak{p}_s$.

Proof. We have

$$\mathfrak{p}_s = \frac{p^2e^2}{pe} = pe = 1-p.$$

and $\lambda_f[G_1 = \varnothing] = \frac{p(1-p)^2}{pe^2} = p^2$ and $\lambda_f[G_1 = \{\langle 0\rangle, \langle 1\rangle\}] = \frac{p^3e^4}{pe^2} = (\overline{p})^2$. Hence

$$\lambda_f[G_1 = J] = (1-p)^2\cdot \mathbf{1}_{J=\{\langle 0\rangle,\langle 1\rangle\}} + \sum_{i=0}^{1} p(1-p)\cdot \mathbf{1}_{J=\{\langle i\rangle\}} + p^2\cdot \mathbf{1}_{J=\varnothing},$$

and so $\lambda_f(\langle i\rangle \in G_1) = (1-p)^2 + p(1-p) = \mathfrak{p}_s$.

Let
$$\lambda_c = \lambda_{1,\gamma}(\cdot \mid [G] \neq \varnothing)$$
be the GW distribution conditional on survival of the whole tree. Let μ_i, μ_f, μ_c be the distribution of the tree G corresponding to the set S under λ_i, λ_f, λ_c, respectively. We define a $\mu_i \times \mu_f \to \mu_c$ measure-preserving map $\psi : 2^{\Omega} \times 2^{\Omega} \to 2^{\Omega}$. The idea is to overlay the two sets S_i, S_f, so that S_i specifies the infinitely extendible part of a tree, and S_f specifies which extensions of finitely extendible strings on the tree are on the tree. So to be precise, we let $\psi(S_i, S_f) = G_i \cup S_f$ where G_i is the tree determined by S_i. By Lemma 4, this gives the correct probability for a finitely extendible string that is the neighbor of an infinitely extendible string to be on the tree. Since each string on a GW-tree is either infinitely extendible or not, it should be clear that ψ is measure-preserving.

A λ_i-ML-random tree may by van Lambalgen's theorem be extended to a λ_c-ML-random tree. To be precise, van Lambalgen's theorem holds in the unit interval $[0,1]$ or equivalently the space 2^{ω}. If (X,μ) is a measure space (we suppress the σ-algebras of measurable sets from the notation) then we call the measure-preserving map $\varphi : (X,\mu) \to ([0,1],\lambda)$, where λ is Lebesgue measure, induced from the measure algebra isomorphism theorem, see Kjos-Hanssen and Nerode [6], the *Carathéodory map* for (X,μ). Using the Carathéodory map we may apply van Lambalgen as desired. We have thus sketched a proof of Theorem 1.

Theorem 1. *For each ML-random BBCDW-tree H there is a ML-random GW-tree G with $[G] = [H]$.*

We next prove that the live part of every infinite ML-random GW-tree is an ML-random BBCDW-tree.

Theorem 2. *For each S, if S is $\lambda_{1,\gamma}$-ML-random then $[G]$ is λ^*_{γ}-random.*

Proof. Suppose $\{U_n\}_{n\in\omega}$ is a λ^*_{γ}-ML-test with $[G] \in \bigcap_n U_n$. Let $\Upsilon_n = \{S : [G] \in U_n\}$. By Lemma 3, $\lambda_{1,\gamma}(\Upsilon_n) = \lambda^*_{\gamma}(U_n)$. Unfortunately, Υ_n is not a Σ^0_1 class, but we can approximate it. While we cannot know if a tree will end up being infinite, we can make a guess that will usually be correct.

Let e be the probability of extinction for a GW-tree. By Lemma 2 we have $e = \frac{\overline{p}}{p}$, so since $p > 1/2$, $e < 1$. Thus there is a computable function $(n,\ell) \mapsto m_{n,\ell}$ such that for all n and ℓ, $m = m_{n,\ell}$ is so large that $e^m \leq 2^{-n}2^{-2\ell}$. Let Φ be a Turing reduction so that $\Phi^G(n,\ell)$, if defined, is the least L such that all the 2^{ℓ} strings of length ℓ either are not on G, or have no descendants on G at level L, or have at least $m_{n,\ell}$ many such descendants. Let
$$W_n = \{S : \text{ for some } \ell,\ \Phi^G(n,\ell) \text{ is undefined}\}.$$

Let $A_G(\ell) = [G] \cap \{0,1\}^{\leq \ell}$ be $[G]$ up to level ℓ. Let the approximation $A_G(\ell, L)$ to $A_G(\ell)$ consist of the nodes of G at level ℓ that have descendants at level L. Let
$$V_n = \{S : A_G(\ell, L) \in U_n \text{ for some } \ell, \text{ where } L = \Phi^G(n,\ell)\}, \text{ and}$$

$$X_n = \{S: \text{ for some } \ell,\ L = \Phi^G(n,\ell) \text{ is defined and } A_G(\ell, L) \neq A_G(\ell)\}.$$

Note

$$\Upsilon_n = \{S : \text{for some } \ell,\ A_G(\ell) \in U_n\ \}$$

Hence

$$\Upsilon_n \subseteq W_n \cup X_n \cup V_n$$

Thus it suffices to show that $\cap_n V_n$, W_n, $\cap_n X_n$ are all $\lambda_{1,\gamma}$-ML-null sets.

Lemma 5. *W_n has $\lambda_{1,\gamma}$-measure zero.*

Proof. If $\Phi(\ell)$ is undefined then there is no L, which means that for the fixed set of strings on G at level ℓ, they don't all either die out or reach m many extensions. But eventually this must happen, so L must exist.

Indeed, fix any string σ on G at level ℓ. Let k be the largest number of descendants that σ has at infinitely many levels $L > \ell$. If $k > 0$ then with probability 1, above each level there is another level where actually $k+1$ many descendants are achieved. So we conclude that either $k = 0$ or k does not exist.

From basic computability theory, W_n is a Σ^0_2 class. Hence each W_n is a Martin-Löf null set.

Lemma 6. *X_n has $\lambda_{1,\gamma}$-probability less than or equal to 2^{-n}.*

Proof. Let E_σ denote the event that all extensions of σ on level L are dead. Let F_σ denote the event that σ has at least m many descendants on $G(L)$.

If $A_G(\ell, L) \neq A_G(\ell)$ then some $\sigma \in \{0,1\}^\ell \cap G$ has at least m many descendants at level L, all of which are dead. If a node σ has at least m descendants, then the chance that all of these are dead, given that they are on G at level L, is $\leq e^m$ (the eventual extinction of one is independent of that of another), hence writing $\mathbb{P} = \mathbb{P}_{1,\gamma}$, we have

$$\mathbb{P}(A_G(\ell, L) \neq A_G(\ell)) \leq \sum_{\sigma\in\{0,1\}^\ell} \mathbb{P}\{E_\sigma \ \& \ F_\sigma\} = \sum_{\sigma\in\{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \cdot \mathbb{P}\{F_\sigma\}$$

$$\leq \sum_{\sigma\in\{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \leq \sum_{\sigma\in\{0,1\}^\ell} e^m \leq \sum_{\sigma\in\{0,1\}^\ell} 2^{-n}2^{-2\ell} = 2^{-n}2^{-\ell}.$$

and hence

$$\mathbb{P}X_n \leq \sum_\ell \mathbb{P}\{A_G(\ell, L) \neq A_G(\ell)\} \leq \sum_\ell 2^{-n}2^{-\ell} = 2^{-n}.$$

X_n is Σ^0_1 since when L is defined, $A_G(\ell)$ is contained in $A_G(\ell, L)$, and $A_G(\ell)$ is co-c.e. in G, which means that if the containment is proper then we will eventually see this. Thus $\cap_n X_n$ is a ML null set.

V_n is clearly Σ^0_1. Moreover $V_n \subseteq \Upsilon_n \cup X_n$, so V_n has probability $\leq 2 \cdot 2^{-n}$, hence $\cap_n V_n$ is a ML null set.

3 Towards a characterization of members of random closed sets

For a set of strings V, let $[V]^{\preceq} = \bigcup\{[\sigma] : \sigma \in V\}$ be the open subset of 2^ω defined by V. For a real number $0 \leq \gamma \leq 1$, the γ-weight $\mathrm{wt}_\gamma(C)$ of a set of strings C is defined by

$$\mathrm{wt}_\gamma(C) = \sum_{w \in C} 2^{-|w|\gamma}.$$

We define several notions of randomness of individual reals.

Definition 1. *A* Martin-Löf γ-test *is a uniformly computably enumerable (c.e.) sequence* $(U_n)_{n<\omega}$ *of sets of strings such that*

$$(\forall n)(\mathrm{wt}_\gamma(U_n) \leq 2^{-n}).$$

A strong ML-γ-test *is a uniformly c.e. sequence* $(U_n)_{n<\omega}$ *such that for each* n *and each prefix-free set of strings* $V_n \subseteq U_n$, $\mathrm{wt}_\gamma(V_n) \leq 2^{-n}$.

A real is (strongly) γ*-random if it does not belong to* $\cap_n[U_n]^{\preceq}$ *for any (strong)* γ*-test* $(U_n)_{n<\omega}$*. If* $\gamma = 1$ *we simply say that the real, or the set of integers* $\{n : x(n) = 1\}$*, is* Martin-Löf random*. For* $\gamma = 1$*, strongness makes no difference.)*

For a measure μ *and a real* x*, we say that* x *is* Hippocrates μ-random *if for each sequence* $(U_n)_{n<\omega}$ *that is uniformly c.e., and where* $\mu[U_n]^{\preceq} \leq 2^{-n}$ *for all* n*, we have* $x \notin \cap_n[U_n]^{\preceq}$*. Let the ultrametric* υ *on* 2^ω *be defined by* $\upsilon(x,y) = 2^{-\min\{n:x(n)\neq y(n)\}}$*. For a measure* μ *on* 2^ω*, we write* $\mu(\sigma)$ *for* $\mu([\sigma])$*.* x *is* Hippocrates γ-energy random *if* x *is Hippocrates* μ*-random with respect to some probability measure* μ *such that*

$$\iint \frac{d\mu(b)d\mu(a)}{\upsilon(a,b)^\gamma} < \infty.$$

Effective Hausdorff dimension was introduced by Lutz [7] and is a notion of partial randomness. For example, if the sequence $x_0x_1x_2\cdots$ is Martin-Löf random, then the sequence $x_00x_10x_20\cdots$ has effective Hausdorff dimension equal to $\frac{1}{2}$. Let $\dim^1_H x$ denote the effective (or constructive) Hausdorff dimension of x; then we have $\dim^1_H(x) = \sup\{\gamma : x \text{ is } \gamma\text{-random}\}$ (Reimann and Stephan [12]).

If $\dim^1_H(x) > \gamma$ then x is Hippocrates γ-energy random [5], and if x is strongly γ-random then x is γ-random and so $\dim^1_H(x) \geq \gamma$.

Kjos-Hanssen obtained a sufficient condition for Membership: each Hippocrates γ-energy random real belongs to a Martin-Löf random closed set under the $(1,\gamma)$-induced distribution.

Theorem 3 ([5])**.** *Each Hippocrates* γ*-energy random real is a* $\textsc{Member}_\gamma$*.*

Here we show a partial converse:

Theorem 4. *Each* $\textsc{Member}_\gamma$ *is strongly* γ*-random.*

Proof. Suppose x is a $\textsc{Member}_\gamma$. Let $\mathbb{P} = \mathbb{P}_{1,\gamma}$ and $p = 2^{-\gamma} \in (\frac{1}{2}, 1]$. Let $i < 2$ and $\sigma \in \Omega$. The probability that the concatenation $\sigma i \in G$ given that $\sigma \in G$ is by definition

$$\mathbb{P}\{\sigma i \in G \mid \sigma \in G\} = p.$$

Hence the absolute probability that σ survives is

$$\mathbb{P}\{\sigma \in G\} = p^{|\sigma|} = \left(2^{-\gamma}\right)^{|\sigma|} = \left(2^{-|\sigma|}\right)^{\gamma}.$$

Let U be any strong γ-test, i.e. a uniformly c.e. sequence $U_n = \{\sigma_{n,i} : i < \omega\}$, such that for all prefix-free subsets $U'_n = \{\sigma'_{n,i} : i < \omega\}$ of U_n,

$$\sum_{i=1}^{\infty} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

Let U'_n be the set of all strings σ in U_n such that no prefix of σ is in U_n. Clearly, U'_n is prefix-free. Let

$$[V_n]^{\preceq} := \{S : \exists i\, \sigma_{n,i} \in G\} \subseteq \{S : \exists i\, \sigma'_{n,i} \in G\}.$$

Clearly $[V_n]^{\preceq}$ is uniformly Σ^0_1. To prove the inclusion: Suppose G contains some $\sigma_{n,i}$. Since G is a tree, it contains the shortest prefix of $\sigma_{n,i}$ that is in U_n, and this string is in U'_n.

Now

$$\mathbb{P}[V_n]^{\preceq} \leq \sum_{i\in\omega} \mathbb{P}\{\sigma'_{n,i} \in G\} = \sum_{i\in\omega} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

Thus V is a test for $\lambda_{1,\gamma}$-ML-randomness. Let S be any $\lambda_{1,\gamma}$-ML-random set with $x \in \Gamma_S$. Then $S \notin \cap_n [V_n]^{\preceq}$, and so $\exists n$, $\Gamma \cap [U_n]^{\preceq} = \varnothing$. Hence $x \notin [U_n]^{\preceq}$. As U was an arbitrary strong γ-test, this shows that x is strongly γ-random.

Theorem 5. *Let $x \in 2^\omega$. We have the implications*

$$dim^1_H(x) > \gamma \;\Rightarrow\; x \textit{ is a } \textsc{Member}_\gamma \;\Rightarrow\; dim^1_H(x) \geq \gamma.$$

Proof. By Reimann [11], each real x with $\dim^1_H(x) > \gamma$ is β-capacitable for some $\beta > \gamma$. By Lemma 2.5 of Kjos-Hanssen [5], x is Hippocrates γ-energy random. The hippocratic restriction that tests are not allowed to consult the measure is not even required, i.e., x is what we may simply call *γ-energy random*. Each strongly γ-random real x satisfies $\dim^1_H(x) \geq \gamma$ (see e.g. Reimann and Stephan [12]).

The second implication does not reverse, as not every real with $\dim^1_H(x) \geq \gamma$ is strongly γ-random (see [12]). We strongly conjecture that the first implication fails to reverse as well, and that there is a Hippocrates γ-energy random real of dimension exactly γ, but will not pursue the matter here.

Our results explain a finding of Barmpalias et al. regarding approximations of $\textsc{Members}$:

Proposition 1. *If x is a real such that the function $n \mapsto x(n)$ is f-computably enumerable for some computable function f for which*

$$\frac{\sum_{j<n} f(i)}{2^{n\gamma}}$$

goes effectively to zero, then x is not γ-random.

Proof. Suppose $n \mapsto x(n)$ is f-c.e. for some such f, and let $F(n) = \sum_{j<n} f(n)$. Let α be any computable function such that $\alpha(n,i) \neq \alpha(n,i+1)$ for at most $f(n)$ many i for each n, and $\lim_{i\to\infty} \alpha(i,n) = x(n)$. Let $c(n,j)$ be the jth such i that is discovered for any $k < n$; so c is a partial recursive function whose domain is contained in $\{(n,j) : j \leq F(n)\}$. For a fixed i, α defines a real α_i by $\alpha_i(n) = \alpha(i,n)$. Let

$$V_n = \{x : \exists j \leq F(n)\ x \upharpoonright n = \alpha_{c(n,j)} \upharpoonright n)\}.$$

Since V_n is the union of at most $F(n)$ many cones $[x \upharpoonright n]$,

$$\mathrm{wt}_\gamma(V_n) \leq \sum_{j=1}^{F(n)} 2^{-n\gamma} = \frac{F(n)}{2^{n\gamma}}$$

Since by assumption $\frac{F(n)}{2^{n\gamma}}$ goes effectively to zero, there is a computable sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\mathrm{wt}_\gamma(U_{n_k}) \leq 2^{-k}$.

Let $U_k = V_{n_k}$. Then U_k is Σ^0_1 uniformly in k, and $x \in \cap_k U_k$. Hence x is not γ-random.

Corollary 1 ([2])**.** *No member of a ML-random closed set under the BBCDW distribution is f-c.e. for any polynomial-bounded f.*

Proof. If f is polynomially bounded then clearly $\frac{\sum_{j<n} f(i)}{2^{n\gamma}}$ goes effectively to zero. Therefore if x is f-c.e., x is not γ-random, hence not a $\textsc{Member}_\gamma$ and thus not a member of a ML-random closed set under the BBCDW distribution.

Computing Brownian slow points. One can show that each non-DNR Turing degree is Low(ML-random, Kurtz random). (A proof of this result, credited to Kjos-Hanssen, is given by Greenberg and Miller [3]. They prove that the converse holds as well.) This can be used to show that each *slow point* (see Mörters and Peres [10]) of any Martin-Löf random Brownian motion must be of DNR Turing degree. [The *fast* points on the other hand form a dense G_δ set, and this can be used to show that there are fast points that are 1-generic and hence do not Turing compute any slow points.] This observation eventually led to the present paper. Rather than working with DNR functions, one could use the work of Hawkes [4] and Lyons [8] to understand random sets that intersect given sets with known probabilities. Positive results for intersection were given in [5], and a negative result is given in Theorem 4 above.

Future work

Conjecture 1. A real x is a MEMBER_γ if and only if x is Hippocrates γ-energy random.

To prove Conjecture 1, one might try to consult a paper of Lyons [9].

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