Enumerability of Strongly Jump-Traceables

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Approximations

Question: As people living in a computable world, how can we get our hands on incomputable objects?

One good answer: Use a computable approximation to the incomputable object.

This talk will focus on four kinds of approximations:

- enumerations of c.e. sets
- the limit lemma for Δ^0_2 sets
- limit approximations which obey cost functions
- traces

- A c.e. set A is a limit $\lim_s A_s$ where the sets are increasing.
- A Δ^0_2 set B is also a limit $\lim_s B_s$, but here the approximation can go back and forth.
- Both c.e. and Δ^0_2 sets can be far from computable, e.g. $\emptyset'$.

Cost functions

An approximation to a Δ^0_2 set A can change a lot, and consequently, the set A can be far from computable.

One way to limit these changes is through obedience to a cost function.

Intuitively, an approximation which obeys a cost function *changes little*. How little depends on the cost function.

Definition

A *cost function* $c(x, s)$ is a computable function from $\mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$.

Intuitively, we think of $c(x, s)$ as the cost payed by an approximation to A to change the approximation to the question $x \in A$ at stage s.

Definition

An approximation A_s *obeys* a cost function c if

$$\sum_s c(x_s, s) < \infty,$$

where x_s is the least x such that $A_s(x) \neq A_s(x-1)$.
A *obeys* c if some approximation of A obeys c.

Cost functions lead to many interesting results, with connections to both randomness and classical computability.

- There is a cost function c_K, sometimes called the "standard cost function", such that A is K-trivial if and only if A obeys c_K. Here

$$c_K(x, s) = \sum_{x<n<s} 2^{-K_s(n)}.$$

  (Nies, 2005)
- For every Δ^0_2 Martin-Löf random set Y, there is a cost function c_Y such that if A obeys c_Y, then $A \leq_T Y$.
- Either of these cost functions can be used to give injury-free solutions to Post's problem. (Kučera, 1986)

Traceability

We now turn to a very different kind of approximation, *traceability*. Instead of approximating the set A directly as a limit, we approximate functions computable from A. Here, approximation means pinning the values of the functions inside small sets.

Definition

An *order* h is a unbounded nondecreasing computable function. A *trace* at order h is a sequence $\langle T_n \rangle$ of finite sets with $|T_n| \leq h(n)$. A function f is *traced* by $\langle T_n \rangle$ if $f(n) \in T_n$ for all n such that $f(n)$ is defined.

We will focus on the case when $\langle T_n \rangle$ is a uniformly c.e. sequence, and will look at traces for partial functions.

Definition

A set A is *h-jump-traceable (h-JT)* if every partial A-computable function is traced by some uniformly c.e. trace at order h. A is *jump-traceable (JT)* if it is h-JT for some order h, and *strongly jump-traceable (SJT)* if it is h-JT for *every* order h.

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The name "jump"-traceable is because the jump function $J^A(e) = \Phi_e^A(e)$ is a universal partial computable function.

Like cost functions, traceability leads to many interesting results, with connections to both randomness and classical computability.

- The class SJT was introduced in the hope it would give a combinatorial characterization of K-triviality. This is not the case, but there is still some hope in that direction.
- Every c.e. strongly jump-traceable set is K-trivial (Cholak Downey Greenberg, 2008).
- Every K-trivial set is jump-traceable.
- Lowness for schnorr randomness is equivalent to being *computably traceable*.
- SJT has connections with Demuth randomness and with diamond classes.

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- Every c.e. strongly jump-traceable set is K-trivial (Cholak Downey Greenberg, 2008). They show $h(n) = \frac{1}{9}\sqrt{\log n}$ suffices.
- Every K-trivial set is jump-traceable. Hölzl, Kräling, and Merkle (2009) show that each K-trivial is jump-traceable for some $h \in O(\log n)$.
- Lowness for schnorr randomness is equivalent to being *computably traceable*.
- SJT has connections with Demuth randomness and with diamond classes.

Some very nice theorems have been proven for c.e. SJTs.

Theorem (Figueira–Nies–Stephan, 2008)

For c.e. sets A, the following are equivalent:

- *A is SJT*
- *A is* strongly superlow*; that is, for every order h, there is a computable approximation to A' with the number of mind-changes bounded by h.*

Theorem (Greenberg–Nies, 2011)

For c.e. sets A, the following are equivalent:

- *A is SJT*
- *A obeys every benign cost function.*

Theorem (Greenberg–Hirschfeldt–Nies)

For c.e. sets A, the following are equivalent:

- *A is SJT*
- *A is below every ω-c.e. Martin-Löf random*
- *A is below every superlow Martin-Löf random*
- *A is below every superhigh Martin-Löf random.*

At this point, there was an obvious question.

Question

To what extent do these results hold for non-c.e. sets?

Some results for JT (as opposed to SJT) suggested that non-c.e. and c.e. sets behaved very differently.

- There is a perfect set of jump-traceable sets.
- For c.e. sets, superlow and jump-traceable coincide. However, there is a superhigh jump-traceable set.

Conversely, Downey and Greenberg were able to show that the K-trivial result they proved with Cholak holds even for non-c.e. SJTs.

Theorem (Downey–Greenberg)

Every SJT set is K-trivial.

Motivated by this result, Downey and Greenberg conjectured that all results for c.e. SJTs would transfer to the non-c.e. case, in the strongest possible way.

Conjecture (Downey–Greenberg)

Every SJT set is below a c.e. SJT set.

There are significant technical hurdles in moving from the c.e. case to the non-c.e. case. Consider the Greenberg–Nies theorem that SJTs are characterized by obeying benign cost functions.

Obeying a cost function means that you have an approximation which changes little.

For c.e. sets, being strongly jump-traceable also means that you have an approximation which changes little. Why?

For Δ^0_2 sets, there is no apparent reason that SJT should give an approximation which changes little.

A Δ^0_2 set can move back and forth, without moving to new places.

By using a "hypercube" argument, it is possible to overcome this difficulty.

Theorem (Diamondstone–Greenberg–Turetsky)

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Theorem (Diamondstone–Greenberg–Turetsky)

Every SJT set is below a c.e. SJT set.

This theorem allows nearly all results about c.e. SJTs to be extended to the general case.

The proof comes in two parts.

Part 1: we show that every SJT set obeys every benign cost function.

Part 2: we construct a single benign cost function c such that if A obeys c, there is a c.e. set $W \geq_T A$ such that for every cost function d, if A obeys d, so does W.

Together, parts 1 and 2 imply the theorem.

Proof.

Let A be an SJT set. By part 1, A obeys every benign cost function, including c. By part 2, there is a c.e. set $W \geq_T A$ which obeys every cost function that A does. In particular, W obeys every benign cost function. By the theorem of Greenberg and Nies, W is SJT. □

Some consequences.

Corollary

For all sets A, the following are equivalent:

- *A is SJT*
- *A is* strongly superlow*.*

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Corollary

If A is SJT, then A is below every superhigh Martin-Löf random.

Thank you