THEORY SPECTRA AND CLASSES OF THEORIES

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ABSTRACT. We analyze the spectra of theories that are ω-stable, theories whose spectra include almost every degree, and theories with uniformly arithmetical n-quantifier fragments. We answer a question from Andrews and Miller [2] by showing that there are ω-stable theories whose spectra are not structure spectra. We show that the spectrum created in Andrews and Knight [1] is not the spectrum of an ω-stable theory, but is the minimal spectrum of any theory with uniformly arithmetical n-quantifier fragments. In addition, we give examples of theory spectra that contain almost every degree, including ones that are known not to be structure spectra.

1. INTRODUCTION

A large body of work in recursive structure theory has been devoted to understanding when a given structure M has a recursive copy, and if it has none, then to classify the set of Turing degrees of copies of M, called the *spectrum* of M. A similar notion is the spectrum of a complete first-order theory, which is the set of Turing degrees of models of the theory. In rare special cases, spectra of theories have been characterized. For example, Solovay [12] characterized the spectra of completions of Peano arithmetic (PA).

Andrews and Miller [2] began the study of comparing the two notions. In particular, they showed that there are spectra of theories that are not spectra of structures and vice versa. All of their examples of spectra of theories that are not spectra of structures had the property that they could not be spectra of atomic theories. They asked first whether there are atomic theories whose spectra are not structure spectra, and second whether there are ω-stable theories whose spectra are not structure spectra. Andrews and Knight [1] answered the first question affirmatively by constructing a theory whose spectrum is exactly the set of degrees of nonstandard models of true arithmetic (TA), and showing that this collection is not a structure spectrum.

We show in Theorem 2.1 that the spectrum constructed by Andrews and Knight is not the spectrum of an ω-stable theory and thus cannot be used to answer the second question. However, we then answer the second question affirmatively (see Section 2.2).

Theorem 2.13. *There is an ω-stable theory T so that $\mathrm{Spec}(T)$ is not a structure spectrum.*

We also examine theory spectra $\mathcal{S}$ with the property that $\mathcal{S}$ has measure 1. If $\mathrm{Spec}(T)$ has measure 1, we say T is an *almost everywhere* theory. This is parallel to the development of the theory of measure 1 structure spectra. Kalimullin showed that $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{b}\}$ is a structure spectrum if $\mathbf{b}$ is low over a computably enumerable set (see [5] and [6] for preliminary results). On the other hand, he produced a $\mathbf{b} \leqslant \mathbf{0}''$ such that $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{b}\}$ is not a structure spectrum [7]. Andrews, Cai, Kalimullin, Lempp and Miller (unpublished) provided more examples by proving that $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{b}\}$ is not a structure spectrum if $\mathbf{b}$ is high or non-GL_2 over $\mathbf{0}'$, so in particular if $\mathbf{b} = \mathbf{0}''$. We show that unlike in the structure spectrum case, $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{b}\}$ is a theory spectrum for any arithmetical $\mathbf{b}$. In fact, we prove the following stronger result.

Theorem 3.7. *Let f be a total recursive function. For each j, let A_j be the arithmetical set with arithmetical index $f(j)$. Then there is a theory T so that $\mathrm{Spec}(T) = \{\mathbf{d} \mid \forall j \, (\mathbf{d} \nleqslant A_j)\}$.*

This, in particular, shows that the collection of non-arithmetical sets is a theory spectrum. It is not known if it is a structure spectrum. This is somewhat surprising since the set of non-hyperarithmetical degrees is a structure spectrum by Greenberg, Montalbán and Slaman [4], but is not a theory spectrum by Andrews and Miller [2].

We show in Theorem 3.1 that every almost everywhere theory T has the property that $T \cap \exists_n$ is uniformly Σ_n^0, and thus there are only countably many almost everywhere theories. We call such theories *Solovay theories*. This leads us to ask:

Question 3.4. *Which are the degrees contained in every almost everywhere theory spectrum?*

Combining results from the literature, we answer the corresponding question for Solovay theories.

Theorem 3.5. *A Turing degree $\mathbf{d}$ is in $\mathrm{Spec}(T)$ for every Solovay theory T if and only if $\mathbf{d}$ computes a presentation of a nonstandard model of TA.*

In particular, the theory constructed by Andrews and Knight [1] is a Solovay theory with minimal spectrum.

1.1. **Abstracting relations.** The following notion of abstracting relations will be necessary below in Sections 2.2 and 3.

In order to answer a question of Hodges and Macintyre about the quantifier complexity of axiomatizations of $\aleph_0$-categorical almost strongly minimal theories, Marker [11] developed a way to add quantifier-complexity to a theory without changing its underlying nature. In particular, he developed what are now called *Marker extensions*, which preserve categoricity, (ω-/super-)stability, and almost strong minimality of a theory while adding quantifiers to the definitions of the important relations.

Given an n-ary relation symbol R with a theory T (in some language $\mathcal{L}$), we have two types of Marker extensions, $T_{\forall,R}$ and $T_{\exists,R}$, defined as follows:

We let T_R' be the following theory:

- $\mathcal{L}'$ is the language obtained by adding to $\mathcal{L}$ two new unary relation symbols U and V and a new $(n+1)$-ary relation symbol S.
- U and V partition the universe into two infinite parts.
- Each symbol of $\mathcal{L}$ only holds on tuples from V, and V is a model of T.

- $S(x, \bar{y}) \to x \in U \wedge \bar{y} \in V^n$.
- $\forall \bar{a} \in V^n \, (R(\bar{a}) \to \exists! \, u \in U \, S(u, \bar{a}))$.
- $\forall \bar{a} \in V^n \, (\neg R(\bar{a}) \to \neg \exists u \in U \, S(u, \bar{a}))$.
- $\forall u \in U \, \exists! \, \bar{a} \in V^n \, (S(u, \bar{a}))$.

T'_R is a complete theory, and $T_{\exists,R}$ is the reduct to the language $\mathcal{L}' \smallsetminus \{R\}$. Note that $R(\bar{x})$ is definable in $T_{\exists,R}$ via the formula $\exists u \in U \, S(u, \bar{x})$. We let $T_{\forall,R}$ be $T_{\exists,\neg R}$. This makes $R(\bar{x})$ $\forall$-definable in $T_{\forall,R}$. If $A \models T_{\exists,R}$ or $A \models T_{\forall,R}$, respectively, and $V^A \cong M$ holds for the expansion to a model of T'_R, then we say that the T-part of A is isomorphic to M.

The two-quantifier $\exists_2$-Marker extension of T for the relation R is the theory $(T_{\forall,R})_{\exists,S}$. The two-quantifier $\forall_2$-Marker extension of T for the relation R is the theory $(T_{\exists,R})_{\forall,S}$. Continuing in this fashion, we can define the n-quantifier $\exists_n$-Marker extension and the $\forall_n$-Marker extension of T for the relation R. Using the natural interpretation, we refer to the T-part of a model of these theories as well.

Lemma 1.1 (Folklore). *Let M be a model of T such that the atomic diagram of M in the language $\mathcal{L} \smallsetminus \{R\}$ is recursive in $\mathbf{d}$. Suppose further that R^M is Σ_n^0 in $\mathbf{d}$. Then the model of the $\exists_n$-Marker extension of T with T-part isomorphic to M is recursively presentable in $\mathbf{d}$.*

Similarly, we can define Marker extensions $T_{(\exists_{n_1}, R_1),(\exists_{n_2}, R_2),\dots}$ for whole collections of relations by simply abstracting each relation R_i to n_i quantifiers as above. The analogous theorem is as follows:

Lemma 1.2 (Folklore). *Let $R_1, \dots$ be a recursive list of relation symbols in $\mathcal{L}$, and let $n_1, \dots$ be a recursive list of positive integers. Let M be a model of T such that the atomic diagram of M in the language $\mathcal{L} \smallsetminus \{R_1, \dots\}$ is recursive in $\mathbf{d}$. Suppose further that each R_i^M is $\Sigma_{n_i}^0$ uniformly in $\mathbf{d}$. Then the model of $T_{(\exists_{n_1}, R_1),(\exists_{n_2}, R_2),\dots}$ with T-part isomorphic to M is recursively presentable in $\mathbf{d}$. In fact, such a $\mathbf{d}$-index can be recursively computed from the index witnessing that each R_i^M is $\Sigma_{n_i}^0$ uniformly in $\mathbf{d}$.*

2. Spectra of ω-stable theories

Andrews and Miller asked two questions, whether there is an atomic theory whose spectrum is not a structure spectrum, and whether there is an ω-stable theory whose spectrum is not a structure spectrum. Andrews and Knight [1] answered the first question by constructing an atomic theory (in fact a completion of PA) whose spectrum is precisely the degrees of nonstandard models of TA. By results of Solovay [12] and Marker [10] on the complexity of models of PA, we know that this is the set of degrees that compute Scott sets containing all arithmetical sets. One attempt to answer the second question of Andrews and Miller would be to show that the same spectrum is also the spectrum of an ω-stable theory. We show that this is not the case, after which we answer the second question via a different construction.

2.1. **The Andrews–Knight Spectrum.**

Theorem 2.1. *The set of degrees of nonstandard models of TA is not the spectrum of an ω-stable theory.*

Proof. We first recall that Andrews and Knight [1, Lemma 15] showed that there is a representation $\mathcal{R}$ of a countable Scott set containing the arithmetical sets and a permutation G of ω, generic over $\mathcal{R}$, so that $\mathcal{R} \oplus G$ computes no representation of the collection of all arithmetical sets. This is proved via forcing to construct a sufficiently generic pair $\mathcal{R}, G$ in a particular forcing partial order $\mathcal{P}$. In particular, $\mathcal{R}$ is a representation of the Scott set $\{X \mid \exists k\, (X \leqslant_T 0^{(k)} \oplus H)\}$ for a fixed Cohen generic H.

Suppose towards a contradiction that T is ω-stable and $\mathrm{Spec}(T)$ is the set of degrees of nonstandard models of TA.

Lemma 2.2. *Let $\mathcal{R}$ be as above and let M be a model of T computable from $\mathcal{R}$. Then there is an $\exists$-type realized in M that is not arithmetical.*

Proof. The proof of Lemma 2.2 follows exactly Andrews and Knight [1, Lemma 13]. Suppose towards a contradiction that all $\exists$-types realized in M are arithmetical. We consider a forcing partial order on pairs of partial permutations of ω as follows: (p_1, p_2) extends (q_1, q_2) if p_1 extends q_1 and p_2 extends q_2. We force in this partial order over M. Since G is generic over $\mathcal{R}$ and thus over M, G can be split into two permutations of ω, G_1 and G_2, so that (G_1, G_2) is generic in this partial order.

We want to show that $G_1(M) \oplus G_2(M)$ computes a representation of the family of arithmetical sets. Let (p_1, p_2) force that $\varphi_{e_1}^{G_1(M)}$ and $\varphi_{e_2}^{G_2(M)}$ are total and are representations of Scott sets containing all arithmetical sets. Call these $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

The following is exactly Andrews and Knight [1, Claim 14].

Claim 2.2.1. $\mathcal{S}_1 \cap \mathcal{S}_2$ *is the collection of all arithmetical sets.*

The key observation in the proof of Claim 2.2.1 is as follows: Let (q_1, q_2) force that a particular column of $\mathcal{S}_1$ is equal to a particular column of $\mathcal{S}_2$. If we view q_i as finite partial maps $\bar{c}_i \in \omega \mapsto \bar{d}_i \in M$, it suffices to know the existential type of $\bar{d}_1$ or $\bar{d}_2$ to compute the column. This is because this existential type declares the possible ways that the quantifier-free diagram of $G_1(M)$ and $G_2(M)$ could be extended. Since these existential types are assumed to be arithmetical, the column in both $\mathcal{S}_1$ and $\mathcal{S}_2$ is arithmetical.

Finally, a computation from $G_1(M) \oplus G_2(M)$ of these two Scott sets gives a computation of a representation of their intersection, thus $G_1(M) \oplus G_2(M)$ computes a representation of the collection of all arithmetical sets. But this contradicts Andrews and Knight [1, Lemma 15]. Thus some $\exists$-type in M is non-arithmetical. $\square$

Lemma 2.3. *Let $\{p_i(\bar{x}) \mid i \in \omega\}$ be a countable collection of non-arithmetical $\exists$-types consistent with T. Let H be Cohen generic over each of the types p_i. Let $\mathcal{R}, G$ be sufficiently generic for the forcing partial order $\mathcal{P}$ that if $W_e^{\mathcal{R}} = p_i$, then some condition forces this. Let M be a model of T computable from $\mathcal{R}$. Then none of the types p_i are realized in M.*

Proof. Suppose $tp_M(\bar{a}) = p_i$. Then some condition forces that $W_e^{\mathcal{R}} = p_i$ for the appropriate index e. But then p_i is arithmetical over the condition, which in turn is arithmetical over H. This contradicts the fact that H is Cohen generic over each of the p_i. $\square$

Suppose, towards a contradiction, that there are only countably many non-arithmetical $\exists$-types. Then Lemma 2.3 gives a model M where none of these

types are realized. By Lemma 2.2, some non-arithmetical $\exists$-type is realized in M, which yields the contradiction. Thus there are uncountably many non-arithmetical $\exists$-types, so uncountably many types, contradicting ω-stability. $\square$

2.2. A spectrum of an ω-stable theory that is not a structure spectrum.

Given an infinite set A and a degree $\mathbf{d}$, we say that A is the range of a limitwise monotonic $\mathbf{d}$-computable function, denoted as $A \in \mathrm{lwm}(\mathbf{d})$, if there is a $\mathbf{d}$-computable function $f(x,s)$ such that $f(x,s) \geqslant f(x,t)$ if $s > t$ (limitwise monotonicity), $\lim_s f(x,s)$ exists for each x, and A is the range of the limit function $\lim_s f(x,s)$, i.e., $A = \{\lim_s f(x,s) \mid x \in \omega\}$.

For any two Δ_2^0-sets A and B, we construct an ω-stable theory whose spectrum is the set of degrees $\mathbf{d}$ so that either $\mathbf{d} \geqslant_T B$ or $A \in \mathrm{lwm}(\mathbf{d})$. We do this by first constructing a pair of prime structures M_A and M_B. The countable saturated model of $\mathrm{Th}(M_A)$ is recursively presentable, while the other models of $\mathrm{Th}(M_A)$ are presentable in any degree $\mathbf{d}$ so that A is the range of a limitwise monotonic $\mathbf{d}$-computable function. The model M_B is recursively presentable, while any presentation of a non-prime model of $\mathrm{Th}(M_B)$ computes B. We then construct a structure M by "gluing" together M_A and M_B so that models of $\mathrm{Th}(M)$ either represent a pair of prime models (M_A, M_B) or a pair of non-prime models (N_A, N_B). In the first case, the presentation computes a presentation of M_A, so it makes A the range of a limitwise monotonic function. In the second case, the presentation presents a nonstandard model of $\mathrm{Th}(M_B)$, so computes B.

We believe that structure spectra should not have a dual nature such as this, but are unable to prove a theorem to this effect. Rather, we choose A and B carefully to ensure that $\mathrm{Spec}(\mathrm{Th}(M))$ is not a structure spectrum.

Lemma 2.4. *Given infinite Δ_2^0-sets A and B, there is an ω-stable structure $M_{A,B}$ such that*
$$\mathrm{Spec}(\mathrm{Th}(M_{A,B})) = \{\mathbf{d} \mid \mathbf{d} \geqslant_T B\} \cup \{\mathbf{d} \mid A \in \mathrm{lwm}(\mathbf{d})\}.$$

Proof. Let $\mathcal{L}_1 = \{E\}$ where E is a binary relation symbol. For $n \in \omega + 1$, we will write K_n for the n-clique. Let M_A be the union of one copy of K_n for each $n \in A$. Then M_A is a prime structure, and the other models of $\mathrm{Th}(M_A)$ are of the form $M_A \cup \bigcup_{i \in I} K_\omega$. Then the countable saturated model of $\mathrm{Th}(M_A)$ is recursively presentable and the other models are presentable in $\mathbf{d}$ if and only if A is the range of a limitwise monotonic function recursive in $\mathbf{d}$. (This is essentially in Khoussainov, Nies and Shore [8].)

Fix a recursive function g so that $B(x) = \lim_s g(x,s)$. Let
$$\mathcal{L}_2 = \{U_i \mid i \in \omega\} \cup \{c_j \mid j \in \omega\},$$
where the U_i are unary relations and the c_j are constant symbols. Let M_B be the structure with universe ω, where c_j is interpreted as the element j and $U_i(j)$ holds if and only if $g(i,j) = 1$. Note that M_B is a prime recursively presentable structure. In any non-prime model $N \equiv M_B$, there is an element x so that $x \neq c_j$ for all c_j. Then $N \models U_i(x)$ if and only if $B(x) = 1$ (as if $\lim_s g(x,s) = k$, then in M_B, $U_j(x) = k$ for all x except for all but finitely many c_j). Thus, any presentation of N must compute B.

Now, we define $\mathcal{L} = \{S, V, W\} \cup L_1 \cup L_2$ where S is a binary relation, and V and W are unary relations. We define the structure M as follows:

- V and W form a partition of M.

- Relations in $\mathcal{L}_1$ hold only on tuples from V and relations in $\mathcal{L}_2$ hold only on elements from W.
- $V^M \cong M_A$ in the language $\mathcal{L}_1$.
- $W^M \cong M_B$ in the language $\mathcal{L}_2$.
- S is the graph of a function from V to W.
- For $n \in A$, the kth element of A, S sends K_n in V to the element c_k.

We claim that M is bi-interpretable with M_A. It follows from the ω-stability of M_A that M is also ω-stable. We interpret M_B as M_A/E (using the fact that E is an equivalence relation, because M_A is a union of cliques). For each symbol U_i, the pre-image in M_A is either a finite union of finite cliques or is a complement of a finite union of finite cliques. Thus they are all definable in M_A. This gives the interpretation of M_B in M_A as M_A/E. The disjoint union of M_A with M_A/E with S as the projection map gives an interpretation I of M in M_A. The obvious interpretation of M_A in M along with I gives a bi-interpretation between M and M_A.

Lastly, we define M' as the $\exists_2$-Marker extension of M for the relation S. Thus M' is bi-interpretable with M. For any $N \equiv M$, we write N' for the bi-interpretable structure that is elementarily equivalent to M'. Since M' is bi-interpretable with M, $\mathrm{Th}(M')$ is also ω-stable, and we will show that $\mathrm{Spec}(\mathrm{Th}(M')) = \{\mathbf{d} \mid \mathbf{d} \geqslant_T B\} \cup \{\mathbf{d} \mid A \in \mathrm{lwm}(\mathbf{d})\}$.

Given any degree $\mathbf{d}$ so that A is the range of a limitwise monotonic function recursive in $\mathbf{d}$, the structure M' is recursive in $\mathbf{d}$. This can be seen since $\mathbf{d}$ presents M_A and M_B and $\mathbf{d}'$ presents S, which, by Lemma 1.2, suffices.

Let N be the structure with

$$V^N \cong M_A \cup \bigcup_{i \in \omega} K_\omega, \text{ and}$$

$$W^N \cong M_B \cup \{x_i \mid i \in \omega\},$$

where S sends V^N to W^N as in M and sends all of K_i to x_i. Note that N is the result of the interpretation I in the structure $M_A \cup \bigcup_{i \in \omega} K_\omega$. Since $M_A \cup \bigcup_{i \in \omega} K_\omega \succeq M_A$, we have $N \succeq M$. N' is recursively presented in B since $V^{N'}$ is recursive, B presents $W^{N'}$, and the map S is recursive in B', which suffices by Lemma 1.2. Thus $\{\mathbf{d} \mid \mathbf{d} \geqslant_T B\} \cup \{\mathbf{d} \mid A \in \mathrm{lwm}(\mathbf{d})\} \subseteq \mathrm{Spec}(\mathrm{Th}(M'))$.

Now, suppose N' is a model of $\mathrm{Th}(M')$ of degree $\mathbf{d}$. If N' is isomorphic to the model M', then N' presents M_A, thus A is the range of a limitwise monotonic function recursive in $\mathbf{d}$. If N' is not isomorphic to the model M', then following the bi-interpretation, we see that $V^{N'} \not\cong M_A$, and therefore there is a copy of K_ω in $V^{N'}$. As $M \models \forall x, y \, (S(x) = S(y) \rightarrow E(x,y))$, we see that the S-image of an element of K_ω is not equal to c_j for any j. Thus $W^{N'}$, and therefore also $\mathbf{d}$, computes B. Thus $\{\mathbf{d} \mid \mathbf{d} \geqslant_T B\} \cup \{\mathbf{d} \mid A \in \mathrm{lwm}(\mathbf{d})\} \supseteq \mathrm{Spec}(\mathrm{Th}(M'))$. $\qquad \square$

We now wish to construct a particular pair of Δ_2^0-sets D and L, where L is in fact low, such that $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$ is not a structure spectrum. In addition, we will construct a third set C so that $\mathbf{c} = \deg(C)$ witnesses that $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$ is not a structure spectrum. In particular, we will show that $\mathbf{c} \notin \mathrm{Spec}(\mathrm{Th}(M_{D,L}))$, but $\mathbf{c} \in \mathcal{S}$ for every structure spectrum $\mathcal{S}$ containing $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$.

The technique we use here for showing that a particular set of Turing degrees is not a structure spectrum involves a property of structures known as the "c.e.

extension property", or "c.e.e.p." If one knows that a structure has this property, it becomes much easier to compute copies of the structure. This can be used to show that a given collection of degrees cannot be a structure spectrum. For example, Andrews and Miller [2] proved that if a structure M has two presentations that form a Σ_1^0-minimal pair, then M has the c.e. extension property, and then used this result to prove that the PA degrees are not a degree spectrum of a structure.

Since graphs are universal for degree spectra, we will phrase the c.e.e.p. in terms of graphs. We use the term "subgraph" to mean "induced subgraph", i.e., "submodel" in the model-theoretic sense. (So, for example, all subgraphs of K_ω are isomorphic to K_ω or K_n for some n.)

Definition 2.5. Fix a computable copy (ω, E) of the random graph. A *code* for a graph G is a subset $A \subseteq \omega$ such that $\Gamma(A) := (A, E \restriction A) \cong G$.

Notation 2.6. We identify subsets of ω with elements of 2^ω. So, for $\alpha \in 2^\omega$ giving the characteristic function for A, $\Gamma(\alpha) := \Gamma(A)$. Given a string $\sigma \in 2^{<\omega}$, we write $\Gamma(\sigma)$ as shorthand for $\Gamma(\sigma 0^\infty)$.

Definition 2.7. Given a graph G, we say that $\sigma \in 2^{<\omega}$ is *consistent with G* if $\Gamma(\sigma)$ is isomorphic to a subgraph of G. Given σ consistent with G and an embedding $f : \Gamma(\sigma) \to G$ (or, equivalently, an isomorphism between $\Gamma(\sigma)$ and a subgraph of G), we say that τ is *consistent with σ, f, G* if $\tau \succ \sigma$ and f extends to an isomorphism between $\Gamma(\tau)$ and a subgraph of G.

Definition 2.8. We say that a graph G has the *c.e. extension property (c.e.e.p.)* if given any σ consistent with G and embedding $f : \Gamma(\sigma) \to G$, the set of all $\tau \in 2^{<\omega}$ such that τ is consistent with σ, f, G is a c.e. set.

The following notion of "c.e.e.p. cover" was introduced by Kalimullin [7]. It is central to our proof, being the way in which we exploit the c.e. extension property.

Definition 2.9. Given $X, C \subseteq \omega$, we say that $C \subseteq \omega^2$ is a *c.e.e.p. cover* for X if for every $e \in \omega$, if Φ_e^X is a code for a graph G with the c.e.e.p., then there is a code A for G such that $A =^* C^{[e]}$ (the eth column of C).

Lemma 2.10. *Let C be a c.e.e.p. cover for X, then every c.e.e.p. graph G presentable by X is also presentable by C.*

Proof. Using the extension property of the random graph, any presentation of a graph G can compute a set A so that $\Gamma(A) \cong G$. Thus if G is presentable by X, then X computes a code for G. If G has the c.e.e.p., then there is some e so that $C^{[e]} =^* A$. Thus $A \leqslant_T C$, so G is presentable by C as well. □

We are now ready to construct L and D so that $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$ is not a structure spectrum. The construction is rather technical, but we start with an arbitrary noncomputable low set L, and use L to construct D and C in such a way that D is Δ_2^0 and C is a c.e.e.p. cover for L, but C does not compute L, nor present D in a limitwise monotonic fashion (i.e., $D \notin \mathrm{lwm}(\deg(C))$). The entire construction is performed by $\mathbf{0}'$, so using the fact that L is low, we may ask L'-questions during the construction.

One unusual feature of our construction is that it is a tree construction, but there is no approximation to the true path. We have a tree T of nodes, which all have different guesses about whether various functions Φ_e^L give codes for graphs with

the c.e.e.p., and, if so, about the corresponding indices of c.e. sets giving consistent finite extensions. However, rather than approximating a true path at stage s, and acting for nodes along this stage s approximation, we instead act for all nodes on T that have been initialized by stage s, and that might possibly end up along the true path. The true path is only defined after the construction is complete, and is then used to define C; both the true path and C will be recursive in $\mathbf{0}'''$. In fact, D is constructed over the course of the "construction", but C is not. Instead, we construct a tree of possible finite approximations to C. Once the construction is complete, the true path through T induces a path through the tree of finite approximations to C, which then defines C.

This is *not* a priority construction, and the nodes on the tree *never injure each other.* Instead, each node acts entirely independently from all the others to define a finite approximation to C, which it then passes to its children when they are activated. At the end of the construction, when the true path is defined, we ignore all of the approximations produced by nodes not along the true path, and integrate the approximations produced by true path nodes, at the stages when they are first activated, to obtain C.

The strategy to make C a c.e.e.p. cover for L acts as a passive global requirement: We are constantly extending a finite approximation to C (or rather, extending the tree of finite approximations), and if Φ_n^L is a code for a graph with the c.e.e.p., then every time we extend the nth column of C, it must be consistent with being a code for that graph (modulo some garbage which never changes). We must also take an active role by constructing the isomorphism, and every so often we must extend the isomorphism we construct.

The strategy to make $L \neq \Phi_e^C$ involves asking L'-questions: When we extend C, we can ask L' whether it is possible to extend in such a way that $\Phi_e^C \neq L$. We must be careful not to violate our passive condition on C when we do this. This is where we take advantage of the c.e.e.p.: We can assume we have a c.e. index for the collection of valid extensions of C, and use that index to phrase the L'-question.

Finally, the strategy to make D not limitwise monotonic in C via f^C involves choosing a follower x, and at each stage s extending C in such a way to force $f^C(x, t)$ to be large, then choosing a new element of D which is not $f^C(x, t)$. In so doing, either we cause $\lambda t. f^C(x, t)$ to increase infinitely often, whence $\lim_t f^C(x, t)$ does not exist, or else we cause $f^C(x, t)$ to become stuck at some largest possible value that is not in D.

Lemma 2.11. *Given a noncomputable low set L, there are sets D and C such that*
  (1) *D is Δ_2^0,*
  (2) *C is Δ_4^0,*
  (3) *C is a c.e.e.p. cover for L,*
  (4) *$L \not\leqslant_T C$, and*
  (5) *D is not the range of any limitwise monotonic function $f \leqslant_T C$.*

Proof. We perform a $\mathbf{0}'$-oracle construction using the tree

$$T = \{(e_i^j)_{i+j<n} \mid n \in \omega, e_i^j \in \omega \cup \{*\}\}.$$

That is, nodes on this tree consist of triangular arrays of elements of $\omega \cup \{*\}$, indexed by i and j, where $i + j < n$. *The entries in the jth column of the array give a succession of guesses for c.e. indices witnessing the c.e.e.p. for $\Gamma(\Phi_j^L)$. A $*$ entry indicates a guess that $\Gamma(\Phi_j^L)$ does not have the c.e.e.p.* The order is just extension

of these triangular arrays; that is, the node $(e_i^j)_{i+j<n}$ is an extension of each of the nodes $(e_i^j)_{i+j<m}$ for $m < n$. We call n the *length* of $\alpha = (e_i^j)_{i+j<n}$, and it is also the number of predecessors of α.

Each node on the tree begins the construction as *inactive*, and at each stage we *activate* finitely many inactive nodes. We may also *cancel* some active nodes, in which case they become inactive (and will never again become active). To each node $\alpha = (e_i^j)_{i+j<n}$, if α begins stage s as active, we have the following data attached to α:

- A finite partial function $g_{\alpha,s} : \omega^2 \to 2$ consisting of *garbage*. (This garbage will be what causes the finite differences between the columns of C and the codes for graphs isomorphic to some $\Gamma(\Phi^L)$.)
- The finite set $E_\alpha = \{j < n \mid (\forall i < n-j) \, e_i^j \neq *\}$. (Note: This is determined entirely by α; we need not even look at the construction to determine E_α.)
- For each $e \in E_\alpha$, a *good* string $\sigma_{\alpha,e}$, together with an isomorphism $\delta_{\alpha,e}$ from $\Gamma(\sigma_{\alpha,e})$ to some subgraph of $\Gamma(\Phi_e^L)$. (This is fixed when α is activated, and thenceforth never changes.)
- For each $e \in E_\alpha$, a *non-bad* string $\tau_{\alpha,e,s}$ consistent with $\sigma_{\alpha,e}, \delta_{\alpha,e}, \Gamma(\Phi_e^L)$ (at least, in the event that Φ_e^L is total).
- We summarize the above by writing $C_{\alpha,s}$, which is also a finite partial function from $\omega^2 \to 2$, given by

$$C_{\alpha,s}(e,x) = \begin{cases} g_{\alpha,s}(e,x) & (e,x) \in \mathrm{dom}(g_{\alpha,s}), \\ \tau_{\alpha,e,s}(x) & (e,x) \notin \mathrm{dom}(g_{\alpha,s}) \text{ and } x \in \mathrm{dom}(\tau_{\alpha,e,s}), \\ \uparrow & \text{otherwise.} \end{cases}$$

- For each of the first s many limitwise monotonic oracle functions f^C, we have an associated follower $x_{\alpha,f}$

Additionally, we have the following global data:

- A sequence $T_0, T_1, T_2, \ldots T_s$ of *thresholds*.
- A sequence $S_0, S_1, S_2, \ldots S_s$ of *markers*.

A note on terminology: Both the "good" and "non-bad" strings with subscript e will be initial segments of codes for graphs isomorphic to $\Gamma(\Phi_e^L)$ in the event that Φ_e^L is total. The difference is that "good" strings come together with partial isomorphisms with $\Gamma(\Phi_e^L)$, while the "non-bad" strings do not come with any particular partial isomorphism attached.

Construction:

At stage 0, we activate the root node with trivial data (each set/function/string is empty).

At stage $s+1$, given the stage s data for all active nodes, we act for each active node, and then activate the least inactive child of each active node. During the action, we need a notion of "α-consistency."

Definition 2.12. Given the data $C_{\alpha,s}$ attached to α at stage s (which we think of as coming with the specifications of what is the garbage g_α, what are the good strings $\sigma_{\alpha,e}$ with associated partial isomorphisms $\delta_{\alpha,e}$, and what are the non-bad strings $\tau_{\alpha,e,s}$), we say that an extension $\hat{C}_{\alpha,s} \supseteq C_{\alpha,s}$ (with associated $\hat{g}_\alpha \supseteq g_\alpha$, $\hat{\sigma}_{\alpha,e} = \sigma_{\alpha,e}$, and $\hat{\tau}_{\alpha,e,s} \supseteq \tau_{\alpha,e,s}$) is *$\alpha$-consistent* if the following hold:

- $\mathrm{dom}(\hat{g}_\alpha) \setminus \mathrm{dom}(g_\alpha)$ is disjoint from $E_\alpha \times \omega$, and

- for each $e \in E_\alpha$, there is an isomorphism of $\Gamma(\hat\tau_{\alpha,j,s})$ with a subgraph of $\Gamma(\Phi_{e,t}^{L_t})$ (for some t) that extends $\delta_{\alpha,e}$.

Note that since L is low, we can $\mathbf{0}'$-computably check, given $C_{\alpha,s}$, whether $\hat C_{\alpha,s}$ is a consistent extension.

When we act, we take five steps. *Steps 1–4 are designed to ensure that D is not limitwise monotonic relative to C, and Step 5 is designed to ensure that C does not compute L. Step 5 also acts to ensure that C is a c.e.e.p. cover for L. However, we also take care when we define $C_{\alpha,s+1}$ in Step 3 that this is consistent with making C a c.e.e.p. cover for L (at least, under the hypothesis that α is on the true path of the construction).*

(1) Pick the new threshold T_{s+1} to be larger than $T_s + k$, where k is the number of active followers $x_{\alpha,f}$, plus the number of active nodes α (which is the number of followers x_{α,f_s} that will be defined in the next step).

(2) For each active α, define a follower x_{α,f_s} where f_s is the sth limitwise monotonic oracle function. This follower should be such that $f_s^X(x_{\alpha,f_s}, t) > T_s$ for some α-consistent extension X of $C_{\alpha,s}$ and some t. (We can $\mathbf{0}'$-computably find such a follower, or determine that no such follower exists. If no such follower exists, we do not choose a follower x_{α,f_s}, as we will not need one.)

(3) For each active α, we will define $C_{\alpha,s+1}$ by extending $C_{\alpha,s}$ a total of $s+1$ times, once for each of the first $s+1$ limitwise monotonic functions:

  - Set $C_{\alpha,s+1}^0 = C_{\alpha,s}$.
  - For each $i \leqslant s$, use the $\mathbf{0}'$-oracle to find an α-consistent extension $C_{\alpha,s+1}^{i+1}$ of $C_{\alpha,s+1}^i$ and a stage t where $f_i^C(x_{\alpha,f_i}, t) > T_{s+1}$, if possible, or where $f_i^C(x_{\alpha,f_i}, t)$ attains the maximum possible value, if $f_i^C(x_{\alpha,f_i}, t) > T_{s+1}$ is impossible. (It may be that "undefined" is the largest possible value, but only if it is impossible to find an α-consistent extension which makes $f_i^C(x_{\alpha,f_i}, t)$ defined.)
  - Set $C_{\alpha,s+1} = C_{\alpha,s+1}^{s+1}$.

(4) Choose S_{s+1} to be the least value larger than T_s which is not any of the largest possible values found in Step 2. (Note that by choice of T_{s+1}, we have $T_s < S_{s+1} < T_{s+1}$.)

(5) For each active α, let β be the least inactive child of α. We activate β. We define the data attached to β as follows:

  - $g_{\beta,s+1} = g_{\alpha,s+1}$.
  - E_β is determined by $\beta = (e_i^j)_{i+j<n}$.
  - Use $0' \equiv_T L'$ to test if there is some finite partial function $\hat C : \omega^2 \to 2$ (with downwards-closed domain) such that $\hat C \supseteq C_{\alpha,s+1}$ and, for each $j \in E_\beta$, if $e_{n-j}^j \neq *$, then the string τ_j (such that $\sigma_{\alpha,j}{}^\frown\tau_j$ extends $\tau_{\alpha,j,s+1}$ and agrees with the jth column of $\hat C$ off of $\mathrm{dom}(g_{\alpha,s+1})$) is in $W_{e_{n-j}^j}$, and for some x,

$$\Phi_n^{\hat C}(x){\downarrow} \neq L(x).$$

    This is not quite the same as a β-consistent extension $\hat C$ where $\Phi_n^{\hat C} \neq L$, because we use the c.e. indices we guessed to measure consistency, rather than directly looking at $\Gamma(\Phi_e^L)$. If there is some such $\hat C$, define

$g_{\beta,s+1}$ to be the union of $g_{\alpha,s+1}$ with the part of $\hat{C}$ in the non-E_β columns, and define $\hat{\sigma}_{\beta,j} = \sigma_{\alpha,j}{}^\frown \tau_j$. Otherwise, define $g_{\beta,s+1} = g_{\alpha,s+1}$ and $\hat{\sigma}_{\beta,j} = \tau_{\alpha,j,s+1}$. (Every $j \in E_\beta$ is also in E_α, so this makes sense, with the exception of $j = n - 1$. One should interpret $\tau_{\alpha,n-1,s+1}$ as the empty string, and $\delta_{\alpha,n-1}$ as the empty function.)

- For each $e \in E_\beta$, we search for a $\sigma \succ \hat{\sigma}_{\beta,e}$ and an isomorphism δ from $\Gamma(\sigma)$ to some subgraph of $\Gamma(\Phi_e^L)$ extending $\delta_{\alpha,e}$ and containing the least element of $\Gamma(\Phi_e^L)$ not in the range of $\delta_{\alpha,e}$. (Using $L' \leqslant_T 0'$, we can check whether such an extension exists.) If such an extension is found, then define $\sigma_{\beta,e} = \sigma$ and define $\delta_{\beta,e} = \delta$. If no such extension exists, then cancel β.
- For each $e \in E_\beta$, we define $\tau_{\beta,e,s+1} = \sigma_{\beta,e}$.
- For each of the first $s + 1$ limitwise monotonic oracle functions f^C, we define $x_{\beta,f} = x_{\alpha,f}$ (provided that α had some follower $x_{\alpha,f}$).

Notice that the way we have defined the stage $s + 1$ data attached to β, we have $C_{\beta,s+1} \supseteq C_{\alpha,s+1}$. (It may or may not be the case that $C_{\beta,t} \supseteq C_{\alpha,t}$ for $t > s + 1$, but this will not matter.)

For each α, let s_α be the stage when α is activated. We define the *true path* Λ inductively by letting the empty string $\lambda \subset \Lambda$, and, given $\alpha = (e_i^j)_{i+j<n} \subset \Lambda$, the *true child* of α is also on Λ, where the *true child* is the node $(e_i^j)_{i+j<n+1}$ for the least c.e. index e_{n+1-j}^j for the set of τ such that $\sigma_{\alpha,j}{}^\frown \tau$ is consistent with $\sigma_{\alpha,j}, \delta_{\alpha,j}, \Phi_j^L$ (if Φ_j^L is total and the set of such τ is c.e.), and is $*$ otherwise. Note that $0''$ can determine whether Φ_j^L is total (using the fact that L is low). If it is total, then L can enumerate the set of α-consistent extensions, and hence $L'' \equiv_T 0''$ can determine, for each e, whether e is a c.e. index for the set of such extensions. Therefore, $0'''$ can determine whether an index exists, and, if so, what the least index is, so can compute the true child of α (uniformly in $\alpha \in \Lambda$), which means Λ is Δ_4^0.

We define $C = \bigcup_{\alpha \in \Lambda} C_{\alpha,s_\alpha}$. By the note at the end of Step 5 of the construction, this union is well defined. The way we have defined C, it is a function from $\omega^2 \to 2$, but we can regard it as a subset of ω in the usual way (via a standard pairing function). Since the true path Λ is Δ_4^0 and since C_{α,s_α} is Δ_2^0 uniformly in α, the set C is also Δ_4^0.

We define $D = \{S_0, S_1, S_2, \ldots\}$. Since D was enumerated in increasing order during the $0'$-computable construction, D is Δ_2^0.

Verification:

When we defined C and D above, we verified that they were Δ_4^0 and Δ_2^0, respectively. We must check that C is a c.e.e.p. cover for L, that C does not compute L, and that D is not limitwise monotonic in C. In order to prove these three things, we will first need two preliminary lemmas.

Claim 2.11.1. *Define α_s to be the longest node along the true path which is active at stage s, and let $C_s = C_{\alpha_s,s}$. Then $C = \bigcup_s C_s$.*

Proof. We simply observe that if β immediately follows α along the true path, and $s_\alpha < s < s_\beta$, then
$$C_{\alpha,s_\alpha} \subseteq C_s = C_{\alpha_s,s} \subseteq C_{\beta,s_\beta}. \qquad \square$$

Claim 2.11.2. *Suppose Φ_e^L is a code for a graph with the c.e. extension property. Then for all $\alpha \in \Lambda$ with $|\alpha| > e$, we have $e \in E_\alpha$.*

Proof. We argue by induction on the length of α that if $\alpha \in \Lambda$ and $|\alpha| > e$, where Φ_e^L is a code for a graph with the c.e.e.p., then $e \in E_\alpha$. This is trivially true for $|\alpha| \leqslant e$. Suppose it is true for α of length at least e, and β is the true child of α. By construction, $\sigma_{\alpha,e}$ is a code for a finite graph isomorphic (via $\delta_{\alpha,e}$) with a subgraph of Φ_e^L, and so the set of all τ such that $\sigma_{\alpha,e}{}^\frown\tau$ is consistent with $\sigma_{\alpha,e}, \delta_{\alpha,e}, \Phi_e^L$ is c.e. This in turn implies that β extends α with a non $*$ term in the eth column. Since either $e \in E_\alpha$ (which means that no $*$ terms appeared in the eth column of α) or $|\alpha| \leqslant e$ (which means α had no eth column), it is also true that no $*$ terms appear in the eth column of β. This means $e \in E_\beta$. $\square$

Claim 2.11.3. *C is a c.e.e.p. cover for L.*

Proof. Suppose Φ_e^L is a code for a graph with the c.e. extension property. By Claim 2.11.2, $e \in E_\alpha$ for all $\alpha \in \Lambda$ with $|\alpha| > e$, so for all but finitely many $\alpha \in \Lambda$ we have defined a string $\sigma_{\alpha,e}$ and a partial isomorphism $\delta_{\alpha,e} : \Gamma(\sigma_{\alpha,e}) \to \Gamma(\Phi_e^L)$, which extend as α moves down the true path. Let $A = \bigcup_{\alpha \in \Lambda} \sigma_{\alpha,e}$, and let $\delta = \bigcup_{\alpha \in \Lambda} \delta_{\alpha,e}$.

By definition, $\sigma_{\alpha,e}$ is the eth column of C_{α,s_α}, modulo a finite initial segment determined by g_{α,s_α}. Moreover, for $\alpha \in \Lambda$ with $e \in E_\alpha$, $g_{\alpha,s}$ is never extended in the eth column, and when a child β of α is activated, it inherits g_{α,s_β} as its g_β, so the finite initial segments in column e determined by g_{α,s_α} are all the same, for all $\alpha \in \Lambda$ with $|\alpha| > e$. Thus A is finitely different from the eth column of C, and δ is an isomorphism from $\Gamma(A)$ to a subgraph of $\Gamma(\Phi_e^L)$. Since the range of $\delta_{\beta,e}$ contains the least element of $\Gamma(\Phi_e^L)$ not in the range of $\delta_{\alpha,e}$ (where α is the parent of β), this subgraph is, in fact, all of $\Gamma(\Phi_e^L)$. This shows that C is a c.e.e.p. cover. $\square$

Claim 2.11.4. *C does not compute L.*

Proof. We will show that $L \neq \Phi_n^C$. Let $\beta \in \Lambda$ be of length n, and let α be β's parent node. During the construction, when β was activated (at stage s_β), we defined C_{β,s_β} to be an extension of C_{α,s_α} such that for each j in E_β, if $e_{n-j}^j \neq *$, then the string τ_j giving the extension in column j from C_{α,s_α} to C_{β,s_β} is in $W_{e_{n-j}^j}$. Moreover, e_{n-j}^j is the least c.e. index for the set of τ such that $\sigma_{\alpha,j}{}^\frown\tau$ is consistent with $\sigma_{\alpha,j}, \delta_{\alpha,j}, \Phi_j^L$, so all future finite extensions $\hat{C}$ where $C_{\beta,s_\beta} \subseteq \hat{C} \subset C$ obey this same constraint.

While satisfying these constraints, we extended C_{α,s_β} in such a way that for some x, we have $L(x) \neq \Phi_n^{C_{\beta,s_\beta}}(x)$ if that was possible, giving us two cases to consider. If it was possible, then $L \neq \Phi_n^C$ and we are done. If it was not possible, then for every x and every finite extension $\hat{C}$ of C_{α,s_β} obeying the constraint, if $\Phi_n^{\hat{C}}(x)\downarrow$, then $\Phi_n^{\hat{C}}(x) = L(x)$. There must be some x_0 such that for all constraint-obeying finite extensions $\hat{C}$, we have $\Phi_n^{\hat{C}}(x)\uparrow$, or else this would give an algorithm for computing L. (Since whether $\hat{C}$ obeys the constraint is determined by the c.e. indices e_i^j, the set of all finite constraint-obeying extensions is c.e.) Thus in this case, $\Phi_n^C(x_0)\uparrow$. So, either way, we obtain $\Phi_n^C \neq L$. $\square$

Claim 2.11.5. *D is not limitwise monotonic in C.*

Proof. We will show that D is not the range of $\lim_s f_e^C(-, s)$. Let α be the longest node along the true path Λ that was active at the beginning of stage e of the construction. There are two possible cases to consider.

Case 1: We did not define any follower x_{α, f_e} at stage e. Because of how we choose followers, that means that for every $t > e$, every $X \supseteq C_{\alpha, e}$ which is α-consistent at stage e, and every x, we have $f_e^X(x, t) \leqslant T_s$. Hence the range of $\lim_s f_e^C(-, s)$ is bounded by T_s. Since D is infinite, it cannot be the range of $\lim_s f_e^C(-, s)$.

Case 2: We defined a follower x_{α, f_e} at stage e. Because of how followers are inherited when new nodes are activated, we have $x_{\beta, f_e} = x_{\alpha, f_e}$ for every $\beta \subset \Lambda$ extending α. Using the approximation $C = \bigcup_s C_s$, when we extended C_s to C_{s+1} at stage $s > e$, we defined C_{s+1} to either force $f_e^{C_{s+1}}(x, t) > T_{s+1}$ for some t (if that was possible), or else to force $f_e^{C_s}(x, t)$ to take the largest possible value otherwise (which must be in the interval (T_s, T_{s+1}) at the first stage s when we were unable to force it above T_s). If the latter ever occurs, the unique value of D in that interval, namely, S_{s+1}, was chosen different from that largest possible value, so $\lim_t f_e^C(x, t) \notin D$. Otherwise, the former occurs at every stage, and hence $\lim_t f_e^C(x, t) = \infty$. $\qquad\square$

This completes the verification of Lemma 2.11. $\qquad\square$

We can now prove the main result of this section.

Theorem 2.13. *There is an ω-stable theory whose spectrum is not the spectrum of any structure.*

Proof. Fix L low noncomputable, and let C and D be as in Lemma 2.11. Let $M_{D,L}$ be as in Lemma 2.4, i.e.,

$$\mathrm{Spec}(\mathrm{Th}(M_{D,L})) = \{\mathbf{e} \mid \mathbf{e} \geqslant_T L\} \cup \{\mathbf{e} \mid D \in \mathrm{lwm}(\mathbf{e})\}.$$

This is an ω-stable theory, and since $L \not\leqslant_T C$ and D is not the range of a limitwise monotonic C-computable function, $\mathbf{c} = \deg C$ is not in $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$. Suppose G is a graph so that $\mathcal{S} = \mathrm{Spec}(G)$ is a structure spectrum containing $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$. Then $\deg(L) \in \mathcal{S}$. It is nontrivial but standard to show that if q has sufficient escape power (relative to D), then D is the range of a limitwise monotonic function computable from q. In particular, if $d(n)$ is defined to be the least stage s such that D_s is stable on an initial segment containing an element of D greater than n, then it is enough that q is infinitely often greater than the function d. Hence if X and Y are sufficiently mutually generic, then $\deg(X)$ and $\deg(Y)$ are both in $\mathcal{S}$ and form a Σ_1^0-*minimal pair*, i.e., any set that is both Σ_1^0 in X and Σ_1^0 in Y is already Σ_1^0, which is ensured by the mutual genericity X and Y. By Andrews and Miller [2, Prop. 3.6], any structure whose spectrum contains two such degrees has the c.e. extension property. In particular, G has the c.e.e.p. Hence C, being a c.e.e.p. cover for L, can compute a copy of G, so $\mathbf{c} \in \mathcal{S}$. Therefore, no structure spectrum can equal $\mathrm{Spec}(\mathrm{Th}(M_{D,L}))$. $\qquad\square$

3. Almost Everywhere Theories

We now show that there are only countably many *almost everywhere theories* (i.e., theories T such that $\mathrm{Spec}(T)$ has measure 1), and that they are all *Solovay theories* (i.e., that $T \cap \exists_n$ is uniformly Σ_n^0). We give several examples of almost everywhere theory spectra, including some known not to be structure spectra. We believe that the following theorem and the lemma in its proof are known, but we include proofs for completeness.

Theorem 3.1. *If T is an almost everywhere theory, then T is a Solovay theory.*

Proof. For an almost everywhere theory T, we fix a recursive function φ_e so that $\{S \mid \varphi_e^S \models T\}$ has positive measure. By the Lebesgue Density Theorem, there is a string σ so that the relative measure of $\{S \mid \varphi_e^S \models T\}$ above σ is greater than $\frac{1}{2}$, i.e., $\mu(\{S \succ \sigma \mid \varphi_e^S \models T\})/\mu(\{S \mid S \succ \sigma\}) > \frac{1}{2}$. By changing e to an index e' where $\Phi_{e'}^S = \Phi_e^{\sigma^\frown S}$, we may assume $\mu(\{S \mid \varphi_e^S \models T\}) > \frac{1}{2}$. Thus, for any $\exists_n$-sentence $\psi := \exists \bar{x}_1 \ldots Q_n \bar{x}_n \psi_0(\bar{x})$,

$$\psi \in T \text{ iff } \mu\left(\{S \mid \mathbb{N} \models \exists \bar{x}_1 \ldots Q_n \bar{x}_n(\varphi_e^S \models \psi_0(\bar{x}))\}\right) > \frac{1}{2}.$$

This can be uniformly translated into a Σ_n^0-question.

Lemma 3.2. *For any arithmetical formula $\varphi(\bar{x}, S)$, there is a formula $\psi(\bar{x})$ of the same arithmetical complexity so that for every $\bar{x}$,*

$$\mathbb{N} \models \psi(\bar{x}) \text{ iff } \mu\left(\{S \mid \mathbb{N} \models \varphi(\bar{x}, S)\}\right) > \frac{1}{2}.$$

Moreover, there is a recursive function mapping φ to ψ.

Proof. We prove the result by induction on the arithmetical complexity of the formula φ. If φ is Σ_1^0, then $\psi(\bar{x})$ is simply

$$\exists m \,\exists \text{ pairwise incomparable } \sigma_1, \ldots \sigma_m \left(\left(\sum_{i \leqslant m} 2^{-|\sigma_i|} > \frac{1}{2}\right) \wedge \bigwedge_{i \leqslant m} \varphi(\bar{x}, \sigma_i)\right).$$

Taking negations gives the result for Π_1^0-formulas.

Now, we prove the result for Σ_n^0-formulas, assuming it holds for Π_{n-1}^0-formulas. Let φ be a Σ_n^0-formula $\exists y_1 \forall y_2 \cdots Q y_n (\varphi_0(\bar{x}, \bar{y}, S))$. We rewrite $\varphi(\bar{x}, S)$ as $\exists m \chi(\bar{x}, m, S)$ where

$$\chi(\bar{x}, m, S) := \bigvee_{y_1 < m} \forall y_2 \cdots Q y_n (\varphi_0(\bar{x}, \bar{y}, S)).$$

Then $\varphi(\bar{x}, S)$ holds for a collection of S of measure $> \frac{1}{2}$ if and only if there is some m so that $\chi(\bar{x}, m, S)$ holds for a collection of S of measure $> \frac{1}{2}$. Using the inductive hypothesis, let $\chi(\bar{x}, m, S)$ holding for a collection of S of measure $> \frac{1}{2}$ be equivalent to a Π_{n-1}^0-formula $\psi_\chi(\bar{x}, m)$. Then $\varphi(\bar{x}, S)$ holds for most S if and only if $\exists m \psi_\chi(\bar{x}, m)$, which is a Σ_n^0-formula. $\square$

This completes the proof of Theorem 3.1. $\square$

It follows from the previous theorem that every almost everywhere theory is computable from $\mathbf{0}^{(\omega)}$, so there can only be countably many such theories:

Corollary 3.3. *There are only countably many almost everywhere theories.* $\square$

Thus there is a set of degrees of measure 1 all of which are contained in every almost everywhere theory spectrum. Without even daring a conjecture, we ask:

Question 3.4. *Which are the degrees contained in every almost everywhere theory spectrum?*

Though we do not know how to characterize the degrees in every almost everywhere theory spectrum, we can characterize the degrees in the spectrum of every Solovay theory. We say that a theory is *uniformly arithmetical* if there is an arithmetical function f such that for each n, if $f(n) = (m, k)$, then $T \cap \exists_n$ is $\Phi_m^{0^{(k)}}$. The

proof shows that the degrees that compute models of every Solovay theory are the same as the degrees that computes models of every uniformly arithmetical theory.

Theorem 3.5. *The following are equivalent for any Turing degree* $\mathbf{d}$:
  (1) $\mathbf{d}$ *is contained in* $\mathrm{Spec}(T)$ *for every uniformly arithmetical theory;*
  (2) $\mathbf{d}$ *is contained in* $\mathrm{Spec}(T)$ *for every Solovay theory;*
  (3) $\mathbf{d}$ *computes a presentation of a nonstandard model of* $\mathrm{Th}(N, +, \cdot)$.

Proof. (1) implies (2) is trivial. (2) implies (3) follows by the construction of Andrews and Knight [1]. In particular, they construct a single Solovay theory T so that $\mathrm{Spec}(T)$ is precisely the degrees computing presentations of nonstandard models of $\mathrm{Th}(N, +, \cdot)$. (3) implies (1) follows by a theorem of Knight [9, Theorem 2.4]. In particular, a theorem of Solovay refined by Marker [9, Corollary 3.3] shows that the degrees that compute presentations of nonstandard models of $\mathrm{Th}(N, +, \cdot)$ are exactly those that compute enumerations of Scott sets containing every arithmetical degree. If X is such an enumeration and T is uniformly arithmetical, then X'' uniformly computes the set of X-indices for $T \cap \exists_n$. Thus, by [9, Theorem 2.4], X computes a model of T. $\qquad\square$

We will now show that, unlike in the case of spectra of structures (cf. Kalimullin [7]), for every arithmetical degree $\mathbf{a}$, $\{\mathbf{d} \mid \mathbf{d} \nleq \mathbf{a}\}$ is the spectrum of a theory. The proof can be extended to show that for any arithmetical sequence of arithmetical sets $(A_i)_{i \in \omega}$, $\{\mathbf{d} \mid \forall i \,(\mathbf{d} \nleq \deg(A_i))\}$ is the spectrum of a theory. Note that any such spectrum has measure 1.

Theorem 3.6. *Let* $\mathbf{a}$ *be an arithmetical degree. Then* $\{\mathbf{d} \mid \mathbf{d} \nleq \mathbf{a}\}$ *is the spectrum of a theory.*

Proof. For a (set A of) degree $\mathbf{a}$, we create a theory T below so that any arithmetical degree $\mathbf{d}$ computes a model of T if and only if $\mathbf{d}$ computes a dense set of elements in 2^ω that are each $\nleq_T A$. The role of the U relations is to code $2^{<\omega}$ into the 1-types of our theory. The role of $(\mathbb{N}, +, \cdot)$ is to be able to define when an element of 2^ω (as represented by the 1-types in the U relations) is $\leqslant_T A$. The V relations enforce that in any model with $(\mathbb{N}, +, \cdot)$ standard, there is no realized element of 2^ω (as represented by the 1-types in the U relations) that is $\leqslant_T A$.

Fix the language
$$\mathcal{L}' := \{U_\sigma(x) \mid \sigma \in 2^{<\omega}\} \cup \{V_i(n, x) \mid i \in \omega\} \cup \{\mathbb{N}(x), +(x, y, z), \cdot(x, y, z)\}.$$

Fix the theory T' generated by the following axioms:
  (1) $U_\emptyset$ and $\mathbb{N}$ form a partition of the universe into two infinite pieces, and $+$ and $\cdot$ hold only on triples from $\mathbb{N}$.
  (2) $\forall n \forall x (V_i(n, x) \rightarrow (\mathbb{N}(n) \wedge U_\emptyset(x)))$.
  (3) $(\mathbb{N}, +, \cdot) \models \mathrm{TA}$.
  (4) For all $\sigma \in 2^{<\omega}$, we have the axioms
$$\forall x \,[\, U_\sigma(x) \leftrightarrow [U_{\sigma 0}(x) \vee U_{\sigma 1}(x)]\,],$$
$$\forall x \,\neg[U_{\sigma 0}(x) \wedge U_{\sigma 1}(x)], \text{ and}$$
$$\exists x \, U_\sigma(x).$$
  (5) For all i so that Φ_i^A is a total $\{0, 1\}$-valued function and for each n, we have an axiom of the form
$$\forall x \,[V_i(n, x) \leftrightarrow U_\sigma(x)]\,,$$

where $|\sigma| = n$ and $\sigma \prec \Phi_i^A$.

(6) For all i so that Φ_i^A is a total $\{0,1\}$-valued function, we have the axiom
$$\forall n \exists^\infty x [V_i(n,x) \wedge \neg V_i(n+1,x)].$$

(7) For all i so that Φ_i^A is a total $\{0,1\}$-valued function, we have the axiom
$$\forall n \forall x \left[V_i(n+1,x) \to V_i(n,x) \right].$$

(8) If i and j are so that $\Phi_i^A = \Phi_j^A$ are total $\{0,1\}$-valued functions, then we have the axiom
$$\forall x \forall n \left[V_i(n,x) \leftrightarrow V_j(n,x) \right].$$

(9) For all i so that Φ_i^A is not a total $\{0,1\}$-valued function, we have
$$\forall n \forall x \neg V_i(n,x).$$

(10) For each i, we have the axiom
$$\forall x \exists n \neg V_i(n,x).$$

First of all, note that Claim 3.6.3 below implies that T' is consistent. Our proof of Theorem 3.6 now proceeds in three claims.

Claim 3.6.1. T' *is a complete theory.*

Proof. Consider two saturated models M and N of T of the same size κ. Certainly $(\mathbb{N},+,\cdot)^M \cong (\mathbb{N},+,\cdot)^N$, as they are both saturated models of the complete theory TA by axiom 3. We need only extend this isomorphism to $U_\emptyset$ as well. By saturation and axiom 4, in both M and N, for each $\rho \in 2^\omega$, there must be κ many elements in $\bigcap_{\sigma \preceq \rho} U_\sigma$. If $\rho \not\leq_T A$, then by axioms 5, 7 and 9, $V_i(n,-)$ holds only for all n below some fixed standard integer, for any element of $\bigcap_{\sigma \preceq \rho} U_\sigma$. Otherwise, fix i such that $\rho = \Phi_i^A$. By axiom 8, the only atomic formulas defining sets nontrivially intersecting $\bigcap_{\sigma \preceq \rho} U_\sigma$ are equivalent to $V_i(n,-)$ for nonstandard n. By axiom 6, there are κ many realizations in both M and N of $V_i(n,x) \wedge \neg V_i(n+1,x)$ for each nonstandard $n \in (\mathbb{N},+,\cdot)^M$ or nonstandard $n \in (\mathbb{N},+,\cdot)^N$, respectively, so we can extend the isomorphism to these. By axioms 7 and 10, these possibilities exhaust all elements of M and N. $\square$

Now fix k so that $\mathbf{a} \leqslant \mathbf{0}^{(k-3)}$. Let T be the theory obtained by abstracting each of the V_i relations by k quantifiers. T is complete, as T' is complete. We claim that $\operatorname{Spec}(T) = \{\mathbf{d} \mid \mathbf{d} \not\leqslant \mathbf{a}\}$.

Claim 3.6.2. $\operatorname{Spec}(T) \subseteq \{\mathbf{d} \mid \mathbf{d} \not\leqslant \mathbf{a}\}$.

Proof. Suppose towards a contradiction that $M \models T$ and $M \leqslant_T A$. Choose an element $b \in U_\emptyset^M$, to which we associate the unique $X \in 2^\omega$ such that for each k, $M \models U_{X \restriction k}(b)$. Note that M computes X. Since M is arithmetical, $(\mathbb{N},+,\cdot)^M$ must be the standard model of TA by Feferman [3]. Thus, axioms 5 and 10 imply that $X \not\leqslant_T A$. Thus M cannot be computable from $\mathbf{a}$. $\square$

Claim 3.6.3. $\operatorname{Spec}(T) \supseteq \{\mathbf{d} \mid \mathbf{d} \not\leqslant \mathbf{a}\}$.

Proof. Let $\mathbf{d}$ be a degree not below $\mathbf{a}$. Then $\mathbf{d}$ uniformly computes a countable dense collection S of reals so that $X \in S$ implies $X \not\leqslant_T A$. Using this, $\mathbf{d}$ computes a structure in the language $\{U_\sigma \mid \sigma \in 2^{<\omega}\} \cup \{\mathbb{N},+,\cdot\}$ where the elements of $U_\emptyset$ are the elements of S, and for all $X \in S$, $U_\sigma(X)$ iff $\sigma \prec X$. By Lemma 1.2, we

only need the correct interpretations of each V_i to be uniformly Σ_k^0 in $\mathbf{d}$ for $\mathbf{d}$ to compute the associated model of T. By axioms 5 and 9, $V_i(n, X)$ holds iff Φ_i^A is a total $\{0, 1\}$-valued function and, if it is, $U_\sigma(X)$ holds, where $\sigma = \Phi_i^A \upharpoonright n$. Using the fact that $A \leqslant \emptyset^{(k-3)}$, each V_i is, in fact, uniformly Π_{k-1}^0 in $\mathbf{d}$. $\square$

This completes the proof of Theorem 3.6. $\square$

Theorem 3.7. *Let f be a total recursive function. For each j, let A_j be the arithmetical set with arithmetical index $f(j)$. Then there is a theory T so that $\mathrm{Spec}(T) = \{\mathbf{d} \mid \forall j \, (\mathbf{d} \nleqslant \deg(A_j))\}$.*

Proof. The proof follows the above construction. The minor required changes are as follows: For each A_j, we again have a sequence of $V_{i,j}$ to enforce that each element is not equal to $\Phi_i^{A_j}$. The function f gives us uniformly an index k_j so that $\mathbf{0}^{(k_j-3)}$ computes A_j. Again, we abstract each of these $V_{i,j}$ relations using a k_j-quantifier abstraction. The rest of the proof remains the same. $\square$

Whether the set of non-arithmetical degrees is the spectrum of a structure is a long-standing open question. The non-hyperarithmetical analog was recently shown to be a structure spectrum by Greenberg, Montalbán and Slaman [4], while it is not a theory spectrum by Andrews and Miller [2]. In contrast to this, the following is an immediate consequence of Theorem 3.7.

Corollary 3.8. *The set of non-arithmetical degrees is a theory spectrum.* $\square$

However, we also have the following:

Corollary 3.9. *A countable intersection of theory spectra need not be a theory spectrum*

Proof. Let $\{\mathbf{a}_i\}_{i\in\omega}$ be a countable anti-chain of arithmetical degrees. Then for each $S \subseteq \omega$, $D_S := \bigcap_{i\in S}\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{a}_i\}$ is a countable intersection of theory spectra. But these form an uncountable family of measure-1 subsets of the Turing degrees. By Corollary 3.3, uncountably many of these are not theory spectra. $\square$

4. Open Questions

We finish with two questions, both left over from Section 3. The first we have already stated explicitly:

Question 3.4. *Which are the degrees contained in every almost everywhere theory spectrum?*

In Theorem 3.6, we showed that every arithmetical degree $\mathbf{a}$ has the property that $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{a}\}$ is a theory spectrum. In relation to this, we ask:

Question 4.1. *What are the degrees $\mathbf{a}$ so that $\{\mathbf{d} \mid \mathbf{d} \nleqslant \mathbf{a}\}$ is a theory spectrum? Are there any non-arithmetical such degrees? Are all such $\mathbf{a}$ at least hyperarithmetical? Note that all such $\mathbf{a}$ are Π_1^1-definable as collections of sets.*

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