LIMIT COMPUTABILITY AND ULTRAFILTERS

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Abstract. We study a class of operators on Turing degrees arising naturally from ultrafilters. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on ω. We can then view a sequence of sets $A = (A_i)_{i\in\omega}$ as an "approximation" of a set B produced by taking the limits of the A_i via $\mathcal{U}$: we set $\lim_{\mathcal{U}}(A) = \{x : \{i : x \in A_i\} \in \mathcal{U}\}$. This can be extended to the Turing degrees, by defining $\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}(A) : A = (A_i)_{i\in\omega} \in \mathbf{a}\}$. The $\delta_{\mathcal{U}}$ — which we call "ultrafilter jumps" — resemble classical limit computability in certain ways. In particular, $\delta_{\mathcal{U}}(\mathbf{a})$ is always a Turing ideal containing $\Delta^0_2(\mathbf{a})$. However, they are also closely tied to Scott sets: $\delta_{\mathcal{U}}(\mathbf{a})$ is always a Scott set containing $\mathbf{a}'$.

Our main result is that the converse also holds: if $\mathcal{S}$ is a countable Scott set containing $\mathbf{a}'$, then there is some ultrafilter $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \mathcal{S}$. We then turn to the problem of controlling the action of an ultrafilter jump $\delta_{\mathcal{U}}$ on two degrees simultaneously, and for example show that there are nontrivial degrees which are is "low" for some ultrafilter jump. Finally, we study the structure on the set of ultrafilters arising from the construction $\mathcal{U} \mapsto \delta_{\mathcal{U}}$; in particular, we introduce a natural preordering on this set and show that it is connected with the classical Rudin-Keisler ordering of ultrafilters. We end by presenting two directions for further research.

Contents

1. Introduction

If X is a set of natural numbers, we can view X as a countable array of sets in a natural way:

Definition 1. *Let $X \subseteq \omega$. Then the we let $X_i = \{j : \langle i, j\rangle \in X\}$ and $X^j = \{i : \langle i, j\rangle \in X\}$ be the ith column and jth row of X, respectively.*

We can then consider the eventual behavior of each row of X. In particular, in case every row of X is finite or cofinite — that is, if $\lim_s(X^j(s))$ exists for every j — then we can define the *limit* of X as

$$\lim(X) = \{j : \lim_s(X^j(s)) = 1\} = \{j : X^j \text{ cofinite}\}.$$

If $Z = \lim(Y)$ and $Y \leq_T X$, we say Z is *limit computable* relative to X.

Shoenfield showed that $A \leq_T X'$ if and only if $A = \lim(Y)$ for some $Y \equiv_T X$. While this is only one of many characterizations of the jump, limit computability is of particular interest because it suggests a wide class of generalizations: given any notion of "generalized limit," we can consider the collection of sets which are *generalized limit computable* relative to a given X. These in turn yield generalized jump operators, that take a set X to the collection of sets which are generalized limit computable relative to X.

In this paper, we investigate limit computability along (nonprincipal) ultrafilters. For each ultrafilter $\mathcal{U}$, we introduce a function $\delta_{\mathcal{U}}$ taking each Turing degree $\mathbf{a}$ to the collection of sets "$\mathcal{U}$-limit computable" in members of $\mathbf{a}$. Besides establishing its basic properties, we characterize the possible values of $\delta_{\mathcal{U}}(\mathbf{a})$, define a notion of "lowness for ultrafilters" and study the question of characterizing these degrees, and examine the ordering on ultrafilters induced by the construction $\mathcal{U} \mapsto \delta_{\mathcal{U}}$.

We recall the definition of an ultrafilter:

Definition 1.1. A set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an *ultrafilter* if it satisfies the following properties:

(1) $\omega \in \mathcal{A}$, $\emptyset \notin \mathcal{A}$.
(2) If $X \in \mathcal{A}$ and $X \subseteq Y \subseteq \omega$, then $Y \in \mathcal{A}$.
(3) If $X, Y \in \mathcal{A}$, then $X \cap Y \in \mathcal{A}$.
(4) For every $X \subseteq \omega$, $X \in \mathcal{A}$ or $(\omega - X) \in \mathcal{A}$.

Additionally, an ultrafilter is *nonprincipal* if it contains no finite set. Although the existence of nonprincipal ultrafilters is not provable in ZF alone, it follows from the axiom of choice that there are $2^{2^{\aleph_0}}$-many ultrafilters on ω, the maximum number possible.

Throughout this paper, we will always write "ultrafilter" to mean "nonprincipal ultrafilter."

Using the fourth ultrafilter axiom, we can take the limit along any ultrafilter of any sequence $(X_i)_{i\in\omega}$ of sets, and for nonprincipal ultrafilters this notion of limit agrees with the classical one when each X_i is finite or cofinite. Taking limits along an ultrafilter then yields the notion of limit computability along an ultrafilter, which in turn yields a class of operators on Turing degrees.

Formally, we proceed as follows. We begin by defining the limit, along an ultrafilter, of an array of reals:

Definition 1.2. For a sequence of sets $X = (X_i)_{i\in\omega}$ and an ultrafilter $\mathcal{U}$, we let

$$\lim_{\mathcal{U}}((X_i)_{i\in\omega}) = \{j : \{i : j \in X_i\} \in \mathcal{U}\}\} = \{j : X^j \in \mathcal{U}\}.$$

Note that, as in the case of classical limit computability, each column X_i functions as an approximation to the limit set $\lim_{\mathcal{U}}(X)$, and dually each row X^j determines the jth bit of $\lim_{\mathcal{U}}(X)$.

We can now define the maps, $\delta_{\mathcal{U}}$:

Definition 1.3. Fix an ultrafilter $\mathcal{U}$. For a Turing degree $\mathbf{a}$, we let

$$\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}((X_i)_{i\in\omega}) : (X_i)_{i\in\omega} = X \in \mathbf{a}\}.$$

Remark 1.4. Note that $\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}((X_i)_{i\in\omega}) : (X_i)_{i\in\omega} = X \leq_T \mathbf{a}\}$: for $X \leq_T \mathbf{a}$, if we fix some set $Y \in \mathbf{a}$, then replacing the 0th column of X by Y results in an array of degree $\mathbf{a}$ whose rows have the same $\mathcal{U}$-limits as those of X.

It is the maps $\delta_{\mathcal{U}}$ and their images, especially $\delta_{\mathcal{U}}(\mathbf{0})$, which are the subject of this article. We call maps of the form $\delta_{\mathcal{U}}$ *ultrafilter jumps.*

We begin by establishing basic closure properties of sets of the form $\delta_{\mathcal{U}}(\mathbf{a})$; this culminates in the following characterization, which is our main result. Recall that a *Scott set* is a collection of reals closed under Turing reducibility and join, and which contains an infinite tree $T \subseteq 2^{<\omega}$ only if it also contains an infinite path through T.

Theorem 1.5. *For a Turing degree* $\mathbf{a}$*, the following are equivalent:*

- $\mathfrak{S}$ *is a countable Scott set containing* $\mathbf{a}'$*.*
- *There is some ultrafilter* $\mathcal{U}$ *such that* $\delta_{\mathcal{U}}(\mathbf{a}) = \mathfrak{S}$*.*

Next, we look at how a single ultrafilter jump can behave with respect to different degrees. We call a degree $\mathbf{a}$ *u-low* if there is some ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$. Using techniques similar to those in the proof of the main theorem, we show the following:

Theorem 1.6. *If* $\mathbf{a}$ *is bounded by a 2-generic or is computably traceable, then* $\mathbf{a}$ *is u-low. Conversely, any degree which computes a DNR*$_2$ *or is high is not u-low.*

We then turn our attention to the structure on the class of all ultrafilters provided by the construction $\mathcal{U} \mapsto \delta_{\mathcal{U}}$. Our main result in this direction is that the partial order induced by this construction is related to a classical reducibility notion on ultrafilters:

Definition 1.7. For ultrafilters $\mathcal{U}, \mathcal{V}$, we write "$\mathcal{U} \leq \mathcal{V}$" if $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all degrees $\mathbf{a}$ on some cone, and "$\mathcal{U} \equiv \mathcal{V}$" if $\mathcal{U} \leq \mathcal{V}$ and $\mathcal{V} \leq \mathcal{U}$.

Theorem 1.8. *The partial order on ultrafilters induced by* $\leq$ *is a quotient of the Rudin-Keisler ordering of ultrafilters on* ω*.*

We also show that the operation of composition of ultrafilter jumps is captured by a binary operation on ultrafilters:

Theorem 1.9. *There is a binary operation* $*$ *such that for every pair of ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$*, we have*

$$\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} = \delta_{\mathcal{U} * \mathcal{V}}.$$

This operation is immediately seen to be compatible with the ordering, $\leq$, so that we have the structure of a partially ordered semigroup.

Finally, we end by presenting two directions for further research. Additionally, throughout this paper we raise a number of questions arising from the theorems above, which remain open.

Throughout this paper, we will need the following pair of basic combinatorial facts:

Definition 1.10. A collection $\{X_i : i \in I\}$ of sets is *free* if every finite Boolean combination is infinite. In particular, each X_i and its complement must be infinite and the X_i must be distinct.

Fact 1.11. *Suppose* $\{X_i : i \in I\}$ *is free, and* $J \subseteq I$*. Then there is an ultrafilter* $\mathcal{U}$ *with* $\{i \in I : X_i \in \mathcal{U}\} = J$*.*

Fact 1.12. *We can effectively find large free sets. Specifically, there is a total* Φ_e *such that*

$$\{\Phi_e^X : X \subseteq \omega\}$$

is a free set.

To prove Fact 1.11, by freeness every finite intersection of elements of $\{X_j : j \in J\} \cup \{\overline{X}_i : i \notin J\}$ is infinite, and so there is an ultrafilter containing $\{X_j : j \in J\} \cup \{\overline{X}_i : i \notin J\}$.

To prove Fact 1.12, construct a computable function $\iota : 2^{<\omega} \to 2^{<\omega}$ which

- builds reals along paths: $\sigma \prec \tau \iff \iota(\sigma) \prec \iota(\tau)$, and

- forces all Boolean combinations to be large: for every $I \subseteq 2^n$, the set
$$\{j : \forall \sigma \in 2^n(\iota(\sigma)(j) = 1 \iff \iota(\sigma) \in I)\}$$
has size at least n.

We then let $\Phi_e^X = \iota(X)$. (Note that, in fact, we have $\Phi_e^X \equiv_T X$.)

Our notation and terminology are mostly standard, except for our notation for rows and columns (see Definition 1). For background on computability theory and set theory, we refer to [DH10] and [Jec03], respectively. For background on ultrafilters, see [CN74].

Finally, a word of reassurance: since ultrafilters usually arise in the context of set theory, it is reasonable to worry that answers to questions about the maps $\delta_\mathcal{U}$ may be independent of ZFC. However, since the action of $\delta_\mathcal{U}$ on a degree $\mathbf{a}$ is determined by countably much information about $\mathcal{U}$, most relevant questions are at worst Π^1_2, and hence set-theoretically absolute (see chapter 25 of [Jec03]). Indeed, with two exceptions, set theory will not be a serious concern in this article. The exceptions are proposition 5.1 — where we examine properties of a natural ordering of ultrafilters arising from the construction $\mathcal{U} \mapsto \delta_\mathcal{U}$ — and section 6.2, where we mention a set-theoretic direction for further research.

2. Basic Properties of $\delta_\mathcal{U}$

In the previous section, we motivated the study of the functions $\delta_\mathcal{U}$ by drawing a comparison with the Turing jump. We begin this section by elaborating on that analogy. The following lemma shows that each function $\delta_\mathcal{U}$ dominates the Turing jump in a completely uniform way:

Lemma 2.1. *There are Turing functionals $\Phi_{e_0}, \Phi_{e_1}, \Phi_{e_2}$ witnessing the following (for every ultrafilter $\mathcal{U}$):*

1. *$\delta_\mathcal{U}$ grows at least as fast as the Turing jump: for every $Y = \lim f(x,s)$, we have $\lim_\mathcal{U}(\Phi^f_{e_0}) = Y$.*
2. *$\delta_\mathcal{U}$ strictly dominates the Turing jump: for every set X, we have $\lim_\mathcal{U}(\Phi^X_{e_1}) \notin \Delta^0_2(X)$.*
3. *For every set X, we have $\lim_\mathcal{U}(\Phi^X_{e_2}) \not\leq_T \lim_\mathcal{U}(X)$, that is, $\delta_\mathcal{U}(deg(X))$ has no top element.*

Proof. (1) follows from the relativized limit lemma. Suppose f is a total X-computable function such that
$$\forall x, \lim_{s\to\infty} f(x,s) \downarrow = Y(x).$$
Let Φ_{e_0} be defined by
$$\Phi^f_{e_0}(\langle i,j\rangle) = f(j,i).$$
Then since $\mathcal{U}$ contains all cofinite sets we have $\lim_\mathcal{U}(\Phi^f_{e_0}) = Y$.

For (2), say that a set Z has the *limit property* if for all j, $\lim_{i\to\infty} Z(\langle i,j\rangle)$ exists. To prove part (ii) we need only construct a $Z \leq_T X$ such that for all nonprincipal $\mathcal{U}$ and all $\hat{Z} \leq_T X$ with the limit property, we have $\lim_\mathcal{U}(Z) \neq \lim_\mathcal{U}(\hat{Z})$. To do this, we proceed as follows. For $e, s \in \omega$, let
$$n_{e,s} = \max\{j : \Phi^X_e(\langle j,e\rangle)[s]\downarrow\}, \quad v_{e,s} = \Phi^X_e(\langle n_{e,s}, e\rangle)$$
(with the convention that $v_{e,s} = 0$ if $n_{e,s}$ is undefined). Now let Z be defined by
$$Z(\langle k,e\rangle) = 1 - v_{e,k},$$
and note that $Z \leq_T X$. The proof of (iii) is completed by noting that whenever Φ^X_e is the characteristic function of a set with the limit property, then
$$\lim_{k\to\infty} Z(\langle k,e\rangle)\downarrow = 1 - \lim_{k\to\infty} \Phi^X_e(\langle k,e\rangle),$$

so $\lim_{\mathcal{U}}(Z)(e) = 1 - \lim_{\mathcal{U}}(\Phi_e^X)(e)$, and hence $\lim_{\mathcal{U}}(Z)$ is not Δ_2^0. This construction, moreover, is effective, so we get the desired index e_1.

The proof of (3) is similar to that of (2). □

Lemma 2.1 raises the problem of classifying the possible images of $\delta_{\mathcal{U}}(\mathbf{a})$.

Lemma 2.2. *$\delta_{\mathcal{U}}(\mathbf{a})$ is a Turing ideal, that is, closed under $\oplus$ and $\leq_T$.*

Proof. Closure under $\oplus$ follows from the fact that

$$\lim_{\mathcal{U}}(\{A_i\}_{i\in\omega}) \oplus \lim_{\mathcal{U}}(\{B_i\}_{i\in\omega}) = \lim_{\mathcal{U}}(\{A_i \oplus B_i\}_{i\in\omega}).$$

To show that $\delta_{\mathcal{U}}(\mathbf{a})$ is closed under $\leq_T$, fix $A = (A_i)_{i\in\omega}$ and suppose $\Phi_e^{\lim_{\mathcal{U}}(A)} = B$. Then let

$$C_i = \{j : \Phi_e^{A_i}(j)[i] \downarrow = 1\}$$

and let $C = (C_i)_{i\in\omega}$. We claim that $\lim_{\mathcal{U}}(C) = B$. To see this, fix $k \in \omega$. There is some initial segment $\sigma \prec \lim_{\mathcal{U}}(A)$ such that $\Phi_e^\sigma(k) \downarrow$; since ultrafilters are closed under finite intersections, for $\mathcal{U}$-many i we have $\sigma \prec A_i$, and for cofinitely many i we have $i > |\sigma|$. Together, these facts imply that for $\mathcal{U}$-many i we have $C_i(k) = \Phi_e^\sigma(k) = B(k)$, which in turn implies $\lim_{\mathcal{U}}(C) = B$. □

In fact, an even stronger closure property is satisfied:

Proposition 2.3. *For every ultrafilter $\mathcal{U}$ and degree $\mathbf{a}$, $\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set. In fact, as in 2.1 this is uniform: there is a single $e \in \omega$ such that for all X and $\mathcal{U}$, we have*

$$\lim_{\mathcal{U}}(X) \text{ is an infinite subtree of } 2^{<\omega} \quad \Rightarrow \quad \lim_{\mathcal{U}}(\Phi_e^X) \text{ is a path through } \lim_{\mathcal{U}}(X).$$

Proof. The intuition behind this proof is that a tree T in $\delta_{\mathcal{U}}(\mathbf{a})$ must be "named" by a sequence of trees $(X_i)_{i\in\omega}$ in $\mathbf{a}$, which — if T is to be infinite — must have arbitrarily long paths. By producing a sequence of increasingly long paths through this sequence of trees, we produce a sequence of sets in $\mathbf{a}$ which $\mathcal{U}$ sends to an infinite path through the named tree. Note that this is intuitively the same argument as for closure under Turing reducibility.

The details are as follows. Suppose $X = (X_i)_{i\in\omega} \in \mathbf{a}$ is such that $T = \lim_{\mathcal{U}}(X)$ is an infinite subtree of $2^{<\omega}$. First, we can assume without loss of generality that each column X_i is also a tree (i.e., downwards closed). To see this, let Y_i be the downwards-closed part of X_i, and let $Y = (Y_i)_{i\in\omega} \in \mathbf{a}$. Since $Y \subseteq X$ we have $\lim_{\mathcal{U}}(Y) \subseteq \lim_{\mathcal{U}}(X)$ — in fact, $\lim_{\mathcal{U}}(Y) = \lim_{\mathcal{U}}(X)$ — and $\lim_{\mathcal{U}}(Y)$ is clearly a tree; so any path we build through $\lim_{\mathcal{U}}(Y)$ will also be a path through T.

So assume X is a sequence of trees. Then X computes a sequence $P = (f_i)_{i\in\omega}$ of sets $f_i \subseteq X_i$ such that f_i is a finite path through X_i of maximal length $\leq i$ (the "$\leq i$" is required to make this search effective). We claim that $\lim_{\mathcal{U}}(P)$ is an infinite path through T.

Clearly $\lim_{\mathcal{U}}(P) \subseteq T$, is closed downwards, and is a path in T (that is, any two elements are comparable); so it is enough to show that $\lim_{\mathcal{U}}(P)$ is infinite. Towards a contradiction, suppose $\sigma \in \lim_{\mathcal{U}}(P)$ of length n is terminal. Then since ultrafilters are closed under finite intersections, we have that for $\mathcal{U}$-many i, $f_i \not\succ \sigma$. Moreover, by definition of P, for all but n-many i, we have

$$|f_i| \leq n \iff ht(X_i) \leq n.$$

Together these imply that for $\mathcal{U}$-many i, X_i has height at most n, and so $\lim_{\mathcal{U}}(X)$ has height at most n as well, which is a contradiction.

By examining the argument above, it is clear that this is a uniform construction, that is, that the construction of P is uniformly computable in X and does not depend on $\mathcal{U}$. □

Remark 2.4. Lemma 2.3 yields an alternate proof of the classical result in reverse mathematics that the theory WKL_0 is strictly weaker than the theory ACA_0 (see chapter VIII of [Sim99]), as follows: via a greedy algorithm we can construct an ultrafilter $\mathcal{U}$ such that the set $\{e : W_e \in \mathcal{U}\}$ is Δ^0_4; this ensures that $\delta_{\mathcal{U}}(\mathbf{0})$ consists entirely of Δ^0_4 sets, and so is not arithmetically closed. This is genuinely different from the standard proof, which follows from iterating the Low Basis Theorem. In particular, neither lowness nor iterated forcing are used in the proof of 2.3.

In the following section, we will show that the converse of 2.3 holds: given any countable Scott set $\mathfrak{S}$ containing $\mathbf{0}'$, there is a nonprincipal ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{0}) = \mathfrak{S}$, and more generally if $\mathbf{a}' \in \mathfrak{S}$ then we can find a $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \mathfrak{S}$.

3. Building Scott sets

We now completely characterize the possible images of ultrafilter jumps by proving the converse of 2.3. This does not provide a characterization of the maps $\delta_{\mathcal{U}}$, however, since we only determine the possible *local* behaviors of those maps. However, in the next section we do make progress towards this goal, by studying what sorts of simultaneous behaviors can be realized by ultrafilter jumps.

Theorem 3.1. *Let* $\mathbf{a}$ *be a degree, and let* $\mathcal{I}$ *be a countable Scott set containing* $\mathbf{a}'$. *Then* $\mathcal{I} = \delta_{\mathcal{U}}(\mathbf{a})$ *for some nonprincipal ultrafilter* $\mathcal{U}$.

Proof of 3.1. Call a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ an *axiom*; informally, we interpret (A, B) as meaning "A is mapped to B by $\lim_{\mathcal{U}}$." Precisely, for C a set of axioms, say that an ultrafilter $\mathcal{U}$ *satisfies* C if $\lim_{\mathcal{U}}(A) = B$ whenever $(A, B) \in C$. Since every family of sets, all of whose finite intersections are infinite, can be extended to a nonprincipal ultrafilter, satisfiability has a purely *combinatorial* definition: if $\mathcal{A} = \{(A_i, B_i) : i \in I\}$ is a set of axioms, we say $\mathcal{A}$ is *consistent* if for every $F \subseteq I$ finite and $n \in \omega$, the intersection

$$[\bigcap_{\substack{j\in F\\ m<n\\ B_j(m)=1}} (A_j)^m] \quad \cap \quad [\bigcap_{\substack{j\in F\\ m<n\\ B_j(m)=0}} \overline{(A_j)^m}]$$

is infinite. Equivalently, $\mathcal{A}$ is consistent if and only if there is a nonprincipal ultrafilter satisfying $\mathcal{A}$.

Remark 3.2. In 4.5 we will consider a different notion of consistency — instead of "A gets mapped to B," our commitments will have the form "A and B get mapped to the same set."

Fix $\mathcal{I} = \{Y_i : i \in \omega\}$, and let $\mathbf{a} = \{X_i : i \in \omega\}$; we will build the desired ultrafilter in stages. We will build a consistent set of axioms C such that (i) for every $A \in \mathbf{a}$ there is some $B \in \mathcal{I}$ with $(A, B) \in C$, and (ii) for every $B \in \mathcal{I}$ there is some $A \in \mathbf{a}$ such that $(A, B) \in C$. We handle (i) at even stages, and (ii) at odd stages:

- In (i), in deciding where to map a set $A \in \mathbf{a}$ we run the risk of contradicting already-enumerated axioms $(A_i, B_i)_{i<k}$ — for example, if the fifth rows of A and A_0 are identical, then the fifth bit of B must be $B_0(5)$. To find a $B \in \mathcal{I}$ to which it is "safe" to map A, it turns out to be equivalent to find a path through a certain infinite binary tree computable in the jump of the (finitely many) axioms built so far.
- In (ii), an apparent difficulty is posed by the fact that $\mathbf{a}$ cannot "see" the commitment we have already made, since right components of axioms lie outside $\mathbf{a}$; however, this turns out not to matter. Suppose $B \in \mathcal{I}$ and $(A_i, B_i)_{i<k}$ is consistent; then if A is "sufficiently different" from the A_is, the set $(A_i, B_i)_{i<k} \cup \{(A, B)\}$ is also consistent. So in deciding what should be mapped to B, we ignore B entirely, and simply choose

some A which is sufficiently different from the sets we have enumerated on the left, so far.

Formally, we proceed as follows:

Even case. Suppose that we have $C_{2s} = \{(A_i, B_i) : i < 2s\}$, and that C_{2s} is consistent, and consider the set $X_s \in \mathbf{a}$. We will find a $B \in \mathcal{I}$ such that $C_{2s} \cup \{(X_s, B)\}$ is consistent. Let $D = \{(A_i)^j : i < 2s, B_i(j) = 1\} \cup \{\mathbb{N} - (A_i)^j : i < 2s, B_i(j) = 0\}$; intuitively, D is the collection of sets we have guaranteed are in the ultrafilter so far. Write $D = \{D_k : k \in \omega\}$, and note that this can be done effectively in $\bigoplus_{i<s} B_i := \hat{B} \in \mathcal{I}$. Say that $\sigma \in 2^{<\omega}$ is *temporarily consistent* if $|\sigma| = n$ and $\forall m < n$,

- $\sigma(m) = 1 \Rightarrow |(X_s)^m \cap (\bigcap_{j<n} D_j)| \geq n$, and
- $\sigma(m) = 0 \Rightarrow |(\mathbb{N} - (X_s)^m) \cap (\bigcap_{j<n} D_j)| \geq n$;

note that $\hat{B} \oplus X_s'$ can uniformly decide whether a $\sigma \in 2^{<\omega}$ is temporarily consistent. Let $T \subseteq 2^{<\omega}$ be the tree of temporarily consistent nodes; since C_{2s} is consistent by induction, T is infinite, and since $\mathcal{I}$ is a Scott ideal containing X_s' and $\hat{B} \in \mathcal{I}$ there is some $B \in \mathcal{I}$ whose characteristic function is a path through T. Then $C_{2s} \cup \{(X_s, B)\}$ is consistent, so let $C_{2s+1} = C_{2s} \cup \{(X_s, B)\}$.

Odd case. Suppose that we have a consistent set of axioms $C_{2s+1} = \{(A_i, B_i) : i < 2s + 1\}$, and consider the set $Y_s \in \mathcal{I}$; we need to find some $A \in \mathbf{a}$ such that $C_{2s+1} \cup \{(A, Y_s)\}$ is consistent. Our main difficulty is that the condition C_{2s+1} we have built so far is not $\mathbf{a}$-computable — in $\mathbf{a}$, we can only see $\{A_i : i < 2s + 1\}$ — so in order to guarantee consistency we will need to ensure that the axiom (A, Y) is consistent with any possible consistent set of axioms with left coordinates from among the A_i $(i < 2s+1)$. To do this, we use a modification of 1.12:

Definition 2. *A set X is* free over *a family of sets $Z = \{Z_i : i \in \omega\}$ if every finite Boolean combination of elements of Z, which is infinite, has infinite intersection with both X and $\omega - X$.*

Lemma 3.3. *We can find free sets in a uniformly effective manner. Specifically, there is an e such that for all $Z = \{Z_i : i \in \omega\}$, Φ_e^Z is free over $\{(Z_i)^j : i, j \in \omega\}$.*

Proof. We need to build X such that for every set B which can be written as a Boolean combination of finitely many elements of Z, either B is finite or both $B \cap X$ and $B \cap \overline{X}$ are infinite. Let $(B_i)_{i\in\omega}$ be a list of all Boolean combinations of elements of Z, with each combination occurring infinitely often, such that for all i, $B_{2i} = B_{2i+1}$; note that such a B can be chosen recursively in Z. At stage 0, set $p_0 = \emptyset$ and say that all i *await attention.* At stage s, suppose we have defined a string $p_s \in 2^{<\omega}$ with length s. Say that j *requires attention* if $j < s$, and at the beginning of stage s, j awaits attention, and $s \in B_j$. Let i be the least number which requires attention, and let $p_{s+1} = p_s^\frown\langle 1\rangle$ if i is even and $p_{s+1} = p_s^\frown\langle 0\rangle$ if i is odd. From now on, say that i *is satisfied*, and move on to stage $s + 1$ - at the beginning of which all j which were satisfied at the beginning of stage s remain satisfied, i is satisfied, and all other requirements await attention.

Let $X = \bigcup p_s$. To see that X has the desired property, first note by induction that for each $j \in \omega$, either B_j is finite or there is some stage s by which j is satisfied. Now, for B a finite Boolean combination of elements of Z which is infinite, let $I = \{j : B_j = B\} = \{j_0, j_0+1, j_1, j_1+1, ...\}$. Each time j_i is satisfied, a new element is added to $B\cap X$; each time j_i+1 is satisfied, a new element is added to $B\cap\overline{X}$. So both $B\cap X$ and $B\cap\overline{X}$ are infinite. □

To finish the proof of Theorem 3.1, we iterate Lemma 3.3 to build an $X \in \mathbf{a}$ such that for each $k \in \omega$, X^k is free over $\{A_i : i < 2s+1\} \cup \{X^j : j < k\}$; we then take $C_{2s+2} = C_{2s+1} \cup \{(A, X)\}$. □

Having completely classified the sets of the form $\delta_{\mathcal{U}}(\mathbf{a})$ in terms of $\mathbf{a}$, we now face the question of classifying ultrafilter jumps themselves:

Question 1. What conditions on a function $f\colon \{\text{Turing degrees}\} \to \{\text{Scott sets}\}$ ensure that $f = \delta_{\mathcal{U}}$ for some $\mathcal{U}$?

One interesting special case is the following:

Question 2. Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed?

This is partly motivated by Remark 2.4, which suggests that there may be further interaction between the study of the maps $\delta_{\mathcal{U}}$ and reverse mathematics.

Currently it is not clear how to approach this type of problem, largely because constructing ultrafilter jumps "to order" is quite difficult. We make some technical progress in this direction, however, in the following section, in which we study what *simultaneous* behaviors can be realized by ultrafilter jumps.

4. Lowness notions

Theorem 3.1 allows us to control the value of $\delta_{\mathcal{U}}(\mathbf{a})$ for a fixed degree $\mathbf{a}$; however, it says nothing about what *simultaneous* behaviors can occur.

First of all, it is obvious that if $\mathbf{b} \geq \mathbf{a}$, then $\delta_{\mathcal{U}}(\mathbf{b}) \supset \delta_{\mathcal{U}}(\mathbf{a})$, and so one particularly interesting question is the following: for what degrees $\mathbf{a}$ is there an ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$? We will call such a degree *u-low*, and we will call a real u-low if it belongs to a u-low degree.

It is easy to see that $\mathbf{0}'$ is not u-low: by a standard diagonalization argument, $\mathbf{0}'$ computes an array $A = (A_i)_{i\in\omega}$ such that the eth row of the eth computable array has no agreement with A_e. More precisely:

Proposition 4.1. *If* $\mathbf{a}$ *contains a* DNR_2 *real, then* $\mathbf{a}$ *is not u-low.*

Proof. Such a degree $\mathbf{a}$ contains a set A such that for every e, if Φ_e is total then

$$\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle).$$

It follows that we can never have $\lim_{\mathcal{U}}(A) = \lim_{\mathcal{U}}(C)$ for any computable array C, so we are done. □

As an aside, note that this rules out the most natural possible positive answer to Question 2:

Corollary 4.2. *No ultrafilter jump* $\delta_{\mathcal{U}}$ *is the "arithmetic closure" operator; that is, for every* $\mathcal{U}$ *there is some* $\mathbf{a}$ *such that* $\delta_{\mathcal{U}}(\mathbf{a}) \neq ARITH(\mathbf{a})$.

However, this does not rule out the existence of ultrafilters which are arithmetically closed in pathological ways, so Question 2 remains open.

In addition, high degrees are not u-low. Recall that a degree $\mathbf{a}$ is high if $\mathbf{a}' \geq \mathbf{0}''$.

Proposition 4.3. *If* $\mathbf{a}$ *is high, then* $\mathbf{a}$ *is not u-low.*

Proof. By Martin's Lemma, such a degree $\mathbf{a}$ computes a *dominant* function f which dominates every computable function. Using f we can compute a set A such that

$$\Phi_e(\langle i, e\rangle) \neq A(\langle i, e\rangle)$$

is true cofinitely often for each $e \in Tot$, i.e., for each e such that Φ_e is total. So as in the DNR_2 case we are done. □

In light of Propositions 4.1 and 4.3, it is reasonable to ask whether *any* nonzero degree is u-low. In fact, many degrees are u-low, including every 2-generic and every computably traceable degree. We begin with a basic combinatorial lemma:

Lemma 4.4. *Suppose $\{A_i : i \in \omega\}$ and $\{X_i : i \in \omega\}$ are collections of sets of natural numbers. Then the following are equivalent:*

1. *There is an ultrafilter $\mathcal{U}$ such that for all i, $\lim_{\mathcal{U}}(A_i) = \lim_{\mathcal{U}}(X_i)$. (Note that here we think of each A_i and X_i as an array of sets, so they will each have their own rows $(A_i)^j, (A_i)^j$ and columns $(A_i)_k$, $(X_i)_k$.)*
2. *For every n, k, there is some $m > k$ such that for every $i, j < n$, we have*

$$(A_i)^j(m) = (X_i)^j(m).$$

If $C = \{(A_i, X_i) : i \in \omega\}$ is a collection of pairs of sets such that the above conditions hold, we call $\mathcal{C}$ a *consistent system*; note that this is a different sense of consistency that that used in 3.1.

Proof. (2) ⇒ (1): Suppose condition (2) holds. Then letting $D_{i,j} = \{x : (A_i)^j(x) = (X_i)^j(x)\}$ be the set on which the jth rows of A_i and X_i agree, we have that $\mathcal{D} = \{D_{i,j} : i, j \in \omega\}$ has the finite intersection principle. Any ultrafilter $\mathcal{U} \supset \mathcal{D}$ witnesses (1), so we are done.

The proof of (1)⇒(2) is similar. □

Theorem 4.5. *Every real bounded by a 2-generic is u-low.*

Recall that a real f is 2-generic if (when viewed as a filter in the poset $2^{<\omega}$) it *meets or avoids* every Σ^0_2 subset of $2^{<\omega}$: if $A \subseteq 2^{<\omega}$ is Σ^0_2 and $f \cap A = \emptyset$, then $\exists \tau \prec f (\forall \sigma \succ \tau, \sigma \notin A)$.

Proof. Fix G 2-generic; we will construct a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(deg(G)) = \delta_{\mathcal{U}}(\mathbf{0})$. Let

$$Tot_G = \{e_0 < e_1 < ...\} = \{e : \Phi^G_e \text{ is total}\}.$$

For $i \in \omega$, let t_i be the first condition in G such that $t_i \Vdash$ "$\Phi^G_{e_j}$ is total" for every $j \leq i$; note that such conditions exist since G is 2-generic. This is the only point in the proof where full 2-genericity is required. (We do not need to take the *least* such conditions, but we do need the t_is to be successively stronger conditions: $t_0 \geq t_1 \geq ....$) Let $\mathbb{P} = \{p_j : j \in \omega\}$ be a listing of Cohen conditions.

We will construct recursive sets X_i such that there is an ultrafilter which maps X_i and $\Phi^G_{e_i}$ to the same set. These X_i will be defined column-by-column, with each column making an increasingly strong guess as to the corresponding column of $\Phi^G_{e_i}$. The complexity of the construction comes from the fact that these guesses must be made effectively, and also must cohere with each other; this second requirement is the reason for having X_i take into account the Φ_{e_j} with $j < i$ in the construction below. Note that the X_i are individually recursive, but the array $(X_i)_{i\in\omega}$ need not be recursive.

Construction 4.6. We define the sets X_i ($i \in \omega$) as follows:

1. For $p_k \not\leq t_i$, the kth column of X_i is empty: $\{\langle j, k\rangle : j \in \omega\} \cap X = \emptyset$.
2. For $p_k \leq t_i$, we define a sequence of conditions $q_0, ..., q_i$ as follows:
   - q_0 is the lexicographically least condition $\leq p_k$ such that
   $$\forall m < k, \Phi^{q_0}_{e_0}(\langle m, k\rangle) \downarrow .$$
   - q_{j+1} is the lexicographically least condition $\leq q_j$ such that
   $$\forall m < k, \Phi^{q_{j+1}}_{e_{j+1}}(\langle m, k\rangle) \downarrow .$$

Note that such q_j exist since $p_k \leq t_i \leq t_{i-1} \leq ... \leq t_0$. We then define the kth column of X_i as by

$$(X_i)_k = \{m : m < k \wedge \Phi_{e_i}^{q_i}(m) \downarrow = 1\}.$$

We claim that there is an ultrafilter $\mathcal{U}$ such that $\lim_{\mathcal{U}}(\Phi_{e_i}^G) = \lim_{\mathcal{U}}(X_i)$ for every i. By Lemma 4.4, it is enough to show that the pair of sequences

$$\{\Phi_{e_i}^G : i \in \omega\}, \{X_i : i \in \omega\}$$

satisfies the property 4.4(1).

To show this, fix n, k, and consider the set of conditions

$$E_{n,m} = \{p \in \mathbb{P} : \exists k > m(\forall i, j < n, (\Phi_{e_i}^p)^j(k) \downarrow = (X_i)^j(k))\}.$$

Each $E_{n,m}$ is Σ_1^0; we will show that G meets each $E_{n,m}$.

It will be enough to show that $E_{n,m}$ is dense below t_n — the 2-genericity of G, together with the fact that $E_{n,m}$ is Σ_1^0, means that G must then meet $E_{n,m}$. Towards this, we fix some condition $p \leq t_n$. There must be some k such that $k > m$ and $p_k \leq p$. Since $p_k \leq p \leq t_n$, the kth column of X_n was constructed according to step (2) of Construction 4.6. Let $q_n \leq p_k$ be the nth condition as defined in the construction of X_n. By the construction of X_n and the fact that $k > m$, we have, for every $i < n$,

$$\Phi_{e_i}^{q_n}(\langle m, k\rangle) = X_i(\langle m, k\rangle);$$

so $q_n \in E_{n,m}$. □

The analogous question for measure remains unsolved.

Question 3. Are sufficiently random reals u-low?

By a similar argument to the proof of Theorem 4.5, we can show that another important computability-theoretic property implies u-lowness:

Theorem 4.7. *Computably traceable implies u-low.*

Recall that a degree $\mathbf{a}$ is computably traceable if for every $h \in \mathbf{a}$, there is a computable j such that $h(n) \in D_{j(n)}$ and $|D_{j(n)}| \leq 2^n$ for every n, where D_e is the canonical finite set coded by e. Note that since there are computably traceable degrees which are not 2-generic and vice versa, Theorems 4.7 and 4.5 compliment each other.

Proof. Let $\{A_i : i \in \omega\}$ be a list of all sets of degree $\leq_T \mathbf{a}$; as in the proof of 4.5, we will construct a collection $\{X_i : i \in \omega\}$ of recursive sets such that there is some ultrafilter $\mathcal{U}$ satisfying $\lim_{\mathcal{U}}(X_i) = \lim_{\mathcal{U}}(A_i)$ for every i. This ultrafilter will then satisfy $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$.

To construct the X_i, we work in stages. Each X_i will have associated with it three functions: the *interval* function f_i, the *block* function g_i, and the *guessing* function h_i. We view A_i and X_i as arrays in the usual way, so that $A_i, X_i \subseteq \omega^2$; in order to construct X_i, we partition the full array ω^2 into "blocks," and partition the nth block into 2^n-many "intervals," and define X_i on each interval separately.

The functions g_i and f_i tell us how to perform this construction: $g_i(n)$ is the number of columns in the nth block, and $f_i(m)$ is the number of columns in the mth interval. (Recall that each block will be partitioned into exponentially-many intervals.) Now we let h_i be a computable function such that for every k, $D_{h_i(k)}$ is a finite set of size 2^k listing the possible behaviors of A_i on the (finitely many) values in the kth block and above the diagonal $\{\langle s, s\rangle : s \in \omega\}$; the existence of such an h_i is guaranteed by the assumption that $\mathbf{a}$ is computably traceable. We then define X_i so that X_i agrees with A_i on at least one interval in each block, by predicting A_i's behavior on the tth interval using the tth element of $D_{h_i(k)}$.

This describes the process for building a single X_i. To ensure that agreements between the X_is and the A_is occur across is, we make intervals of X_{i+1} correspond to blocks of A_i; this guarantees that the collection of pairs $\{(A_i, X_i) : i \in \omega\}$ forms a consistent system (see 4.4).

Precisely, the construction is the following:

- At stage 0 we have $f_0 \colon x \mapsto 1$ and $g_0 \colon x \mapsto 2^x$.
- At stage $i+1$, blocks from stage i become intervals and the new blocks are exponentially large collections of intervals. That is, $f_{i+1} = g_i$, $g_{i+1}(0) = f_{i+1}(0)$, and

$$g_{i+1}(n+1) = \sum_{j=2^{n+1}-1}^{2^{n+2}-1} f_{i+1}(j).$$

The h_i are then computable maps such that for every x, (the canonical code for) the finite set

$$A_i \upharpoonright \{\langle m, n\rangle : m \leq n \text{ and } \sum_{t=0}^{x-1} g_i(t) \leq n < \sum_{t=0}^{x} g_i(t)\}$$

is an element of $D_{h_i(x)} = \{s_1^{i,x} < s_2^{i,x} < \ldots < s_{2^x}^{i,x}\}$.

We then let X_i be defined by copying the set coded by $s_m^{i,x}$ on the mth interval in the xth block. It is easy to see that for every j, there is at least one interval in the jth block such that X_i and A_i agree on the first j-many rows. Since our construction nests blocks at level i inside intervals at level $i+1$, it is now easy to see by Lemma 4.4 that the family $\{(A_i, X_i) : i \in \omega\}$ is consistent. □

The above results provide a wide swath of u-low degrees. However, our knowledge of u-lowness is still very incomplete. The following question remains open:

Question 4. Is there an exact characterization of u-lowness in terms of classical computability-theoretic properties?

Less ambitiously, note that no Δ^0_2 degree is either 2-generic or computably traceable, and so the following question remains open:

Question 5. Is there a nonzero Δ^0_2 real which is u-low?

Finally, while investigating u-lowness, an even stronger notion of weakness with respect to ultrafilters arises. We say a degree $\mathbf{a}$ is *u-trivial* if $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$ for *every* ultrafilter $\mathcal{U}$.

Question 6. Is there a nonzero u-trivial degree?

Any u-trivial degree must be low: if X is not low, then there is a Scott set containing $\emptyset'$ and not containing X', and by Theorem 3.1 there is an ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(deg(X)) \neq \delta_{\mathcal{U}}(REC)$. In particular, a positive answer to Question 6 would yield a strong positive answer to 5.

5. Comparing ultrafilters

We now turn to what the construction of the maps $\delta_{\mathcal{U}}$ can tell us about the set of ultrafilters. We begin by defining a natural preorder arising from these maps, and then turn to a natural associated algebraic (semigroup) structure; we end by presenting a connection with a classical structure on ultrafilters, the *Rudin-Keisler order*. This section is self-contained, but for background and further information on the Rudin-Keisler order, and orderings on ultrafilters in general, see [CN74], especially chapters 9 and 16.

Definition 3. *For $\mathcal{U}, \mathcal{V}$ ultrafilters, let $\mathcal{U} \leq \mathcal{V}$ if for some degree $\mathbf{b}$, we have $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all $\mathbf{a} \geq_T \mathbf{b}$; that is, $\mathcal{U} \leq \mathcal{V}$ if $\delta_{\mathcal{V}}$ dominates $\delta_{\mathcal{U}}$ on a cone. We write $\mathcal{D}_{ult}$ for the resulting partial order on (equivalence classes of) ultrafilters.*

Note that $\mathcal{U} < \mathcal{V}$ does *not* imply that, on a cone, $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$. Indeed, it is not clear whether such a situation ever occurs.

Question 7. Are there $\mathcal{U}, \mathcal{V}$ such that $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$ for all $\mathbf{a}$ (on a cone)?

Proposition 5.1. *$\mathcal{D}_{ult}$ is* ω_1-directed*: given any ω_1-sized collection $\{\mathcal{U}_\eta : \eta \in \omega_1\}$ of ultrafilters, there is a $\mathcal{V}$ with $\mathcal{U}_\eta < \mathcal{V}$ for every η.*

Proof. We use Fact 1.12 to construct an ultrafilter $\mathcal{V}$ which dominates each $\mathcal{U}_\eta$ on a cone. Let $h : \mathbb{R} \to \omega_1$ be a function such that for each $\alpha \in \omega_1$, the set $\{r : h(r) > \alpha\}$ contains a cone, and which is *Turing invariant*: $r \equiv_T s \Rightarrow h(r) = h(t)$. For example, we could take $h\colon r \mapsto \omega_1^r$.

Now fix, for each real r, a real $\hat{r}$ such that $\hat{r} \geq_T s$ for every $s \in \delta_{\mathcal{U}_\eta}(deg(r))$ with $\eta < h(r)$. Using 1.12 we can construct an ultrafilter $\mathcal{V}$ such that $\hat{r} \in \mathcal{V}(deg(r))$ for every real r. This $\mathcal{V}$ dominates each $\mathcal{U}_\eta$ on a cone, so we have $\mathcal{U}_\eta < \mathcal{V}$ for every $\eta \in \omega_1$. □

Note that this argument cannot be easily extended to give upper bounds of larger sets of ultrafilters. Indeed, it is consistent that there are exactly ω_2-many ultrafilters on ω, in which case ω_1-directedness is the most we could hope for.

Question 8. What can be said about $|\mathcal{D}_{ult}|$? (Note that we have $2^{\aleph_0} < |\mathcal{D}_{ult}| \leq 2^{2^{\aleph_0}}$; the second inequality is trivial, and the first follows from an argument similar to that of Proposition 5.1. Moreover, it is consistent — and follows from GCH— that $|\mathcal{D}_{ult}| = 2^{2^{\aleph_0}}$.)

Additionally, the proof of Proposition 5.1 says nothing about the *optimality* of the upper bound constructed.

Question 9. What sets of ultrafilters have least upper bounds in $\mathcal{D}_{ult}$?

Note that it is not even clear whether *finite* sets of ultrafilters have least upper bounds.

We now show that the set of ultrafilters carries a natural semigroup structure which is compatible with the degree structure $\mathcal{D}_{ult}$:

Definition 4. *For $\mathcal{U}, \mathcal{V}$ ultrafilters, let*

$$\mathcal{U} * \mathcal{V} = \{X : \{b : \{a : \langle a, b\rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

It is clear that $\mathcal{U} * \mathcal{V}$ is again an ultrafilter, and that the operation $*$ is associative. The crucial property of $*$ is the following:

Proposition 5.2. *For all $\mathcal{U}, \mathcal{V}$ we have $\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} = \delta_{\mathcal{U}*\mathcal{V}}$.*

Proof. For a set X, let

$$X^\sharp = \{\langle\langle i, j\rangle, k\rangle : \langle i, \langle j, k\rangle\rangle \in X\}.$$

We claim that $\lim_{\mathcal{U}*\mathcal{V}}(X^\sharp) = \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X))$, as follows:

$$x \in \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X)) \iff \{j : \langle j, x\rangle \in \lim_{\mathcal{V}}(X)\} \in \mathcal{U} \iff \{j : \{k : \langle k, \langle j, x\rangle\rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}$$

$$\iff \{j : \{k : \langle\langle k, j\rangle, x\rangle \in X^\sharp\} \in \mathcal{V}\} \in \mathcal{U} \iff \{\langle k, j\rangle : \langle\langle k, j\rangle, x\rangle \in X^\sharp\} \in \mathcal{U}*\mathcal{V} \iff x \in \lim_{\mathcal{U}*\mathcal{V}}(X^\sharp).$$

Since the operation $^\sharp$ is invertible and preserves Turing degree, we have shown that $\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}}(\mathbf{a}) = \delta_{\mathcal{U}*\mathcal{V}}(\mathbf{a})$ for every degree $\mathbf{a}$. □

Remark 5.3. Note that Proposition 5.2 only holds on the level of *degrees*: in general, given ultrafilters $\mathcal{U}$ and $\mathcal{V}$ and a set X there need be no ultrafilter $\mathcal{W}$ with $\lim_{\mathcal{W}}(X) = \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X))$. For example, take $X = (X_i)_{i\in\omega}$ with $X_0 = \omega$ and $X_i = \emptyset$ for $i > 0$. Then $\lim_{\mathcal{W}}(X) = \{0\}$ and $\lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X)) = \emptyset$ regardless of the choice of $\mathcal{U}, \mathcal{V}, \mathcal{W}$.

Proposition 5.2 immediately yields:

Corollary 5.4. *The operation $*$ is compatible with $\mathcal{D}_{ult}$: if $\mathcal{U}_0 \le \mathcal{U}_1$ and $\mathcal{V}_0 \le \mathcal{V}_1$, then $\mathcal{U}_0 * \mathcal{V}_0 \le \mathcal{U}_1 * \mathcal{V}_1$. Moreover, $*$ is well-defined on elements of $\mathcal{D}_{ult}$.*

Proposition 5.2 also provides us with a "jump" operator on $\mathcal{D}_{ult}$:

Definition 5. *For $\mathcal{U}$ an ultrafilter, let $\mathcal{U}' = \mathcal{U} * \mathcal{U}$.*

Proposition 5.5. *For every $\mathcal{U}$ we have $\mathcal{U} < \mathcal{U}'$.*

Proof. This is a refinement of Corollary 4.2. That $\mathcal{U} \le \mathcal{U}'$ is immediate. To show that this is strict, fix a sufficiently large degree $\mathbf{a}$ and suppose $\mathcal{U}' \le \mathcal{U}$. Then we have (using Lemma 2.1(1) for the first equality)

$$\delta_{\mathcal{U}}(\mathbf{a}') \subseteq \delta_{\mathcal{U}'}(\mathbf{a}) \subseteq \delta_{\mathcal{U}}(\mathbf{a}),$$

contradicting the relativized version of Proposition 4.1. □

This natural algebraic structure, compatible with the preorder, suggests that $\mathcal{D}_{ult}$ may be an interesting degree structure in its own right. We end by providing further evidence for this: a connection between $\mathcal{D}_{ult}$ and a more classical ordering of ultrafilters, the *Rudin-Keisler ordering*:

Definition 6. *For $\mathcal{U}, \mathcal{V}$ ultrafilters, $\mathcal{U}$ is* Rudin-Keisler reducible to *$\mathcal{V}$ — and we write $\mathcal{U} \le_{RK} \mathcal{V}$ — if for some $f\colon \omega \to \omega$ we have*

$$\mathcal{U} = f^{-1}(\mathcal{V}), \text{ that is, } X \in \mathcal{V} \iff f^{-1}(X) \in \mathcal{U}.$$

We write $\mathcal{U} \le^f_{RK} \mathcal{V}$ if f witnesses $\mathcal{U} \le_{RK} \mathcal{V}$.

The connection between Rudin-Keisler reducibility and our $\mathcal{D}_{ult}$ is provided by the following:

Theorem 5.6. *Suppose $\mathcal{U} \le^f_{RK} \mathcal{V}$. Then if $f \le_T \mathbf{a}$, we have $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$.*

Proof. Given $X = (X_i)_{i\in\omega} \in \mathbf{a}$, define $Y_i = \{n : n \in X_{f(i)}\}$, $\quad Y = (Y_i)_{i\in\omega}$. Now by our assumption on f we have

$$n \in \lim_{\mathcal{V}}(Y) \iff \{i : n \in X_{f(i)}\} \in \mathcal{V} \iff \{i : n \in X_i\} \in \mathcal{U} \iff n \in \lim_{\mathcal{U}}(X).$$

But this means $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$, so we are done. □

Corollary 5.7. *If $\mathcal{U} \le_{RK} \mathcal{V}$, then $\mathcal{U} \le \mathcal{V}$.*

Given this connection between $\mathcal{D}_{ult}$ and the Rudin-Keisler ordering, it is natural to ask:

Question 10. Is there a characterization of $\le$ in terms of combinatorial properties of ultrafilters? In particular, does $\le^*$ coincide with $\le_{RK}$?

6. Further directions

We end by presenting two directions for further research.

6.1. **Filter jumps.** We have investigated maps $\delta_{\mathcal{U}}$ for $\mathcal{U}$ an ultrafilter. However, this construction applies equally well to filters:

Definition 6.1. A *filter* is a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\omega)$ satisfying conditions (1)-(3) of definition 1.1. For $\mathcal{F}$ a filter and $A = (A_i)_{i\in\omega}$ a sequence of sets, set $\lim_{\mathcal{F}}(A) = \{i : A_i \in \mathcal{F}\}$; then for $\mathbf{a}$ a Turing ideal, define

$$\delta_{\mathcal{F}}(\mathbf{a}) = \{\lim_{\mathcal{F}}(A) : A \leq_T \mathbf{a}\}.$$

To preserve the analogy with limit computability, we will restrict our attention to filters containing $\mathcal{F}$.

Intuitively, this is a more "biased" notion of limit computability, since it is in general easier to have $X \notin \mathcal{F}$ than to have $X \in \mathcal{F}$. This is reflected in the fact that, in general, the resulting "filter jumps" $\delta_{\mathcal{F}}$ — while they may correspond to natural computability-theoretic operations — do not always yield Turing ideals. For example, $\delta_{\mathcal{F}_{cof}}(\mathbf{a}) = \Sigma^0_2(\mathbf{a})$, which is not closed under Turing reduction. On the positive side, note that $\delta_{\mathcal{F}}(\mathbf{a})$ is always closed under $\oplus$, and the limit lemma immediately implies that $\delta_{\mathcal{F}}(\mathbf{a}) \supseteq \Delta^0_2(\mathbf{a})$. Beyond this, however, it seems difficult to establish how these more general operations behave, and so the question of characterizing the possible images of filter jumps, in analogy with Theorem 3.1, is open:

Question 11. Fix a Turing degree $\mathbf{a}$. For what classes $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is there some filter $\mathcal{F}$ with $\delta_{\mathcal{F}}(\mathbf{a}) = \mathcal{I}$?

In particular, ensuring closure under Turing reducibility appears difficult.

Question 12. What filters $\mathcal{F}$ have the property that $\delta_{\mathcal{F}}(\mathbf{a})$ is a Turing ideal for all $\mathbf{a}$?

Moving on to trying to control the action of $\delta_{\mathcal{F}}$, we can (as in section 4) define a degree $\mathbf{a}$ to be f*-low* if there is a filter $\mathcal{F}$ such that $\delta_{\mathcal{F}}(\mathbf{a}) = \delta_{\mathcal{F}}(\mathbf{0})$. Clearly f-lowness is implied by u-lowness, and DNR_2 degrees are not f-low.

Question 13. Which degrees are f-low?

6.2. **Ultrafilter jumps of Turing ideals.** Our definition of $\delta_{\mathcal{U}}$ makes sense, not just for degrees, but for Turing ideals:

Definition 6.2. For $\mathfrak{I}$ a countable Turing ideal, let $\delta_{\mathcal{U}}(\mathfrak{I}) = \{\lim_{\mathcal{U}}(A) : A \in \mathfrak{I}\}$.

Then we can develop the theory of ultrafilter jumps in this broader context. By and large, the resulting picture is the same. Most importantly, by essentially the same proof as Theorem 3.1, we obtain:

Corollary 6.3. *Suppose $\mathfrak{I}$ is a countable Turing ideal and $\mathfrak{K} \supseteq \mathfrak{I}$. Then the following are equivalent:*

- *$\mathfrak{K}$ is a countable Scott set containing $\mathbf{a}'$ for every $\mathbf{a} \in \mathfrak{I}$.*
- *There is an ultrafilter $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{K}$.*

There are, however, slight differences. For example, note that when generalized to ideals, Question 7 has a simple negative answer:

Proposition 6.4. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters, and fix a Turing ideal $\mathfrak{I}$. Then there is an ideal $\mathfrak{K} \supseteq \mathfrak{I}$ such that $\delta_{\mathcal{U}}(\mathfrak{K}) = \mathfrak{K} = \delta_{\mathcal{V}}(\mathfrak{K})$.*

Proof. We alternately apply $\delta_{\mathcal{U}}$ and $\delta_{\mathcal{V}}$ infinitely many times to $\mathfrak{I}$. Let $\mathfrak{I}_0 = \mathfrak{I}$, $\mathfrak{I}_{n+1} = \delta_{\mathcal{U}*\mathcal{V}}(\mathfrak{I}_n)$, and let

$$\mathfrak{K} = \bigcup_{i\in\omega} \mathfrak{I}_n.$$

It is easy to check that $\mathfrak{K}$ satisfies the desired properties. □

Having already generalized to countable Turing ideals, we can further consider the question of characterizing $\delta_{\mathcal{U}}(\mathfrak{I})$ for *uncountable* Turing ideals $\mathfrak{I}$. To a large extent, the possible behavior of ultrafilter jumps on uncountable ideals is already determined by their possible behavior on countable ideals, and even on individual degrees. However, the proof of Theorem 3.1 relied on enumerating the Turing ideals in question, and so breaks down as soon as we pass to uncountable ideals. This raises the question of whether our characterization still holds for uncountable ideals, and furthermore, to what extent the answer to this question depends on the axioms of set theory.

In general, this is unknown. However, at least a certain amount of set-theoretic independence does occur. By Theorem 3.1, if $\mathfrak{I}$ is countable and arithmetically closed then there is some ultrafilter $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{I}$. This can fail to generalize to uncountable ideals in a strong way. Write "$\mathfrak{I} \prec 2^\omega$" if the structure $(\omega, \mathfrak{I}; 0, 1, +, \times, \in)$ is an elementary substructure of $(\omega, 2^\omega; 0, 1, +, \times, \in)$. For example, if $\mathfrak{I} \prec 2^\omega$ then $\mathfrak{I}$ is arithmetically closed, since "$X = Y'$" is definable in the language of second-order arithmetic. Then we have:

Theorem 6.5 (Schweber)**.** *It is independent of ZFC whether elementary subideals of 2^ω are always fixed by some $\delta_{\mathcal{U}}$:*

- *Consistently with ZFC (in fact, provably from V=L), for any $\mathfrak{I} \prec 2^\omega$ there is some ultrafilter with $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{I}$.*
- *If V satisfies projective determinacy PD, then there is a forcing extension W of V with the same reals as V and a $\mathfrak{I} \in W$ such that $\mathfrak{I} \prec 2^\omega$ but no ultrafilter $\mathcal{U}$ satisfies $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{I}$.*

The proof of Theorem 6.5 will appear in the fourth author's Ph.D. thesis. However, note that this result still falls far short of understanding the extension of Theorem 3.1 to uncountable Turing ideals.

Question 14. Is it consistent with ZFC that every *arithmetically closed* Turing ideal is fixed by some $\delta_{\mathcal{U}}$?

Question 15. Does ZFC+PD *prove* that there is an $\mathfrak{I} \prec 2^\omega$ not fixed by any $\delta_{\mathcal{U}}$?

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