Will the dog catch the duck?

This is a solution to the "Will the dog catch the duck?" puzzle from `https://fivethirtyeight.com/features/will-the-dog-catch-the-duck/`.

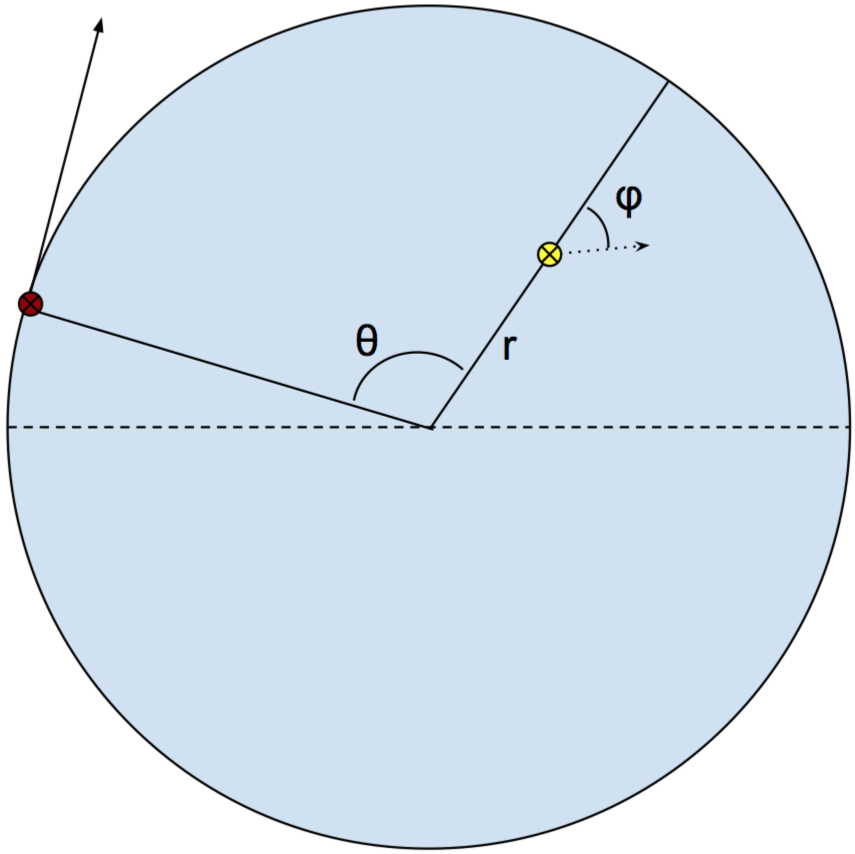
Answer. *The dog needs to go at least V times as fast as the duck to catch it, where V is the unique positive zero of the function*

$$f(v) = \sqrt{v^2 - 1} - \pi - \arccos \frac{1}{v}.$$

(According to WolframAlpha, this is approximately 4.6033.)

Proof. Working in units where the radius of the pond is 1 and the speed of the duck is 1, let $r(t)$ be the distance from the center of the pond to the duck to the shoreline at time t, and let $\theta(t)$ be the angular separation between the dog and the duck (which is also the distance along the shoreline). The duck escapes if, for some t, $r(t) = 1$ and $\theta(t) > 0$.

Diagram:



In which direction should the duck swim at time t? In other words, in the diagram, what should the angle φ be? The duck has two competing goals: he wants to maximize his progress towards the shore, which is expressed by the derivative $r'(t)$, and simultaneously minimize the dog's progress towards the closest point along the shore, which is $-\theta'(t)$. When the duck swims as in the diagram, we can see that $r'(t) = \cos\varphi$. If the dog were standing still, we would have $\theta'(t) = \frac{\sin\varphi}{r}$, but since the dog is running along the shore at some speed v, we actually have

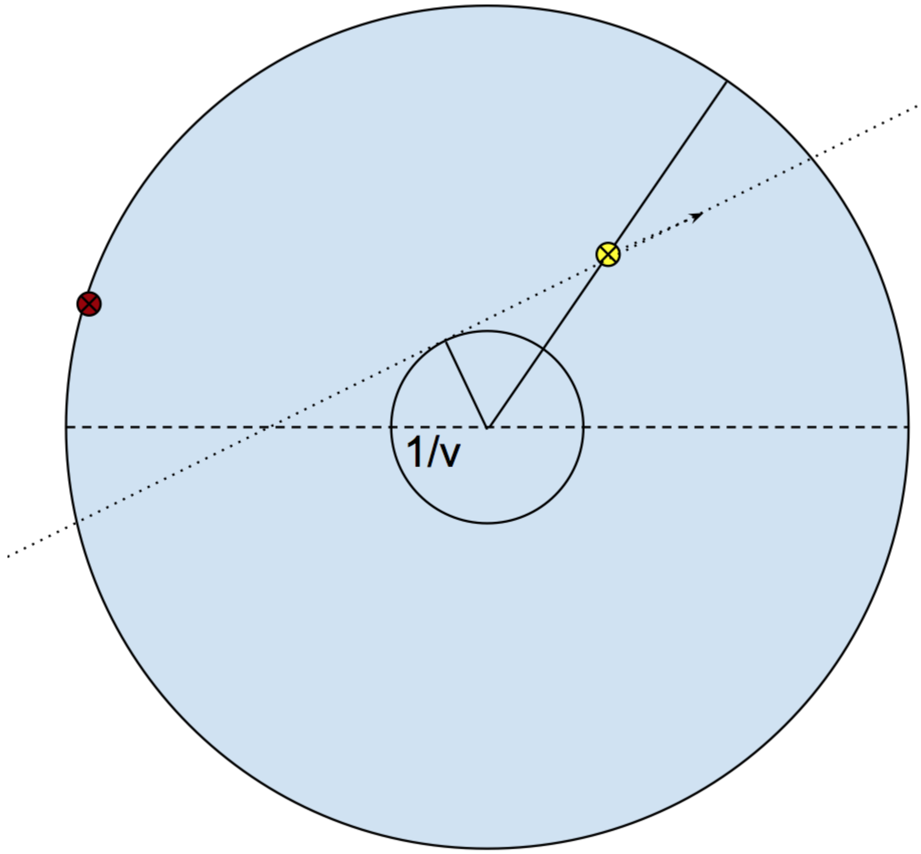
$$\theta'(t) = \frac{\sin\varphi}{r} - v.$$

If $r < 1/v$, the duck can swim at an angle $\varphi = \arcsin rv$, which implies $\theta'(t) = 0$. In other words, the duck can make progress towards the shore without the dog making any progress towards that closest point. In practice, that means the duck can reach the circle at radius $1/v$ and still be opposite the dog. After leaving this circle, $1/r < v$, which means $\theta'(t) < 0$, so the dog always makes some progress towards her goal as the duck makes progress towards his. How should the duck travel? In practice it's only $\frac{dr}{d\theta}$ which matters, the ratio of the duck's progress to the dog's, so the duck should swim at an angle φ which maximizes

$$\frac{dr}{d\theta} = \frac{r'(t)}{-\theta'(t)} = \frac{\cos\varphi}{(\sin\varphi)/r - v}.$$

By the first derivative test, this is maximized when $\varphi = \arcsin\frac{1}{rv}$.

There's a geometric interpretation of this: the duck travels along one of the two lines through his position that is tangent to the circle at radius $1/v$. To see this, note that $\sin\varphi = \frac{1}{rv}$ is opposite / hypotenuse for the right triangle in the diagram below.



This means that after the duck reaches the circle at radius $1/v$, he moves in a straight line (only changing directions if the dog does). This should not be surprising, since if the duck were not going straight, he would be able to take a more efficient path. If both animals follow the optimal paths, the overall motion is that the duck first travels to the circle at radius $1/v$ while keeping the dog directly opposite, and then travels along the line tangent to that circle while the dog travels *the long way around* to the point where the duck meets the circle. It might seem surprising that the dog travels the long way around, but in fact the dog *is* always traveling the most direct route to the closest point on the shoreline to the duck (except when the duck first crosses the circle when both are equal). If the dog started traveling the other way the duck would immediately switch to traveling along the other tangent line.

The total distance the duck needs to travel from the circle of radius $1/v$

to the shore is $\sqrt{1-\frac{1}{v^2}}$, and the total distance the dog needs to travel is $\pi + \arccos\frac{1}{v}$, so the duck escapes if

$$\sqrt{1-\frac{1}{v^2}} < \frac{1}{v}\left(\pi + \arccos\frac{1}{v}\right)$$

and the dog catches the duck if

$$\sqrt{1-\frac{1}{v^2}} \geq \frac{1}{v}\left(\pi + \arccos\frac{1}{v}\right).$$

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