Low LR upper bounds

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Background

Definition

A *Martin-Löf test* is a sequence $(U_n)_{n\in\omega}$ of uniformly Σ^0_1 classes of reals such that for all n, we have $\mu(U_n) \leq 2^{-n}$.

A real X *passes* a Martin-Löf test (U_n) if $X \notin \bigcap_n U_n$.

A real X is *Martin-Löf random* if it passes every Martin-Löf test.

Notes:

- There is a universal test.
- We can relativize Martin-Löf randomness by relativizing tests in the obvious way.
- We write $\mathcal{MLR}$ for the class of Martin-Löf random reals. We write $\mathcal{MLR}^X$ for the class of reals which are ML-random relative to X.
- ML-randomness has an alternative characterization in terms of the prefix-free Kolmogorov complexity K.

In randomness, lowness notions play an important role. Many have been introduced:

- low-for-random: $\mathcal{MLR}^A = \mathcal{MLR}$
- low-for-K: $K \leq^+ K^A$
- K-trivial: $K(A\restriction n) \leq^+ K(n)$
- ...

Remarkably, these lowness notions are all equivalent.

From the point of view of randomness, these sets frequently play the role that the computable sets play in computability theory.

Weak Reducibilities

Definition

A preorder $\leq_*$ on the power set of the natural numbers is called a *weak reducibility* if it is weaker than Turing reducibility, i.e. $A \leq_T B$ implies $A \leq_* B$.

Weak reducibilities are often associated to lowness notions by relativizing (or partially relativizing) the lowness notion.

$\leq_{LR}$

- The *LR reducibility* is the weak reducibility defined by $A \leq_{LR} B$ if $\mathcal{MLR}^B \subseteq \mathcal{MLR}^A$.
- The bottom degree $\{A \mid A \leq_{LR} 0\}$ is the set of low-for-random reals.
- $A \leq_{LR} B$ is different from A low-for-random relative to B; actually A is low-for-random relative to B iff $A \oplus B \leq_{LR} B$.
- A straightforward relativization does not give a transitive relation.

Think about LR reducibility using the contrapositive definition: $A \leq_{LR} B$ iff $\overline{\mathcal{MLR}}^A \subseteq \overline{\mathcal{MLR}}^B$. So $A \leq_{LR} B$ if B has at least as much power as A to derandomize, i.e. to compute Martin-Löf tests covering any given real.

The *LR* reducibility is therefore a covering notion: $A \leq_{LR} B$ if the universal ML-test relative to B covers the universal ML-test relative to A.

Because LR is a covering notion, it relates to the reverse mathematics of measure theory. It has connections with other definitions which have arisen in that context:

- B is *almost-everywhere dominating* if, for almost all reals X (w.r.t. Lebesgue measure), every X-computable function is dominated by some B-computable function.
- B is *uniformly almost-everywhere dominating* if there is a single B-computable function f such that for almost all X, every X-computable function is dominated by f.

Theorem (Kjos-Hanssen/Miller/Solomon 2006)

The following are equivalent for a real B:

- *B is almost-everywhere dominating*
- *B is uniformly almost-everywhere dominating*
- $0' \leq_{LR} B$.

A useful corollary: if $0' \leq_{LR} B$, then B is high.

Proof of corollary: If f dominates every X-computable function, then f dominates every computable function. So if B is uniformly almost-everywhere dominating, it computes a dominant function.

Observation

There are reals A, B such that $A \leq B$, but $A \oplus B \nleq_{LR} B$.

Proof.

Let A be a promptly simple K-trivial. Let B be a low set such that $A \oplus B \equiv_T 0'$. Then $A \leq_{LR} 0 \leq_{LR} B$. However, since B is low, it is not uniformly almost-everywhere dominating, so $A \oplus B \nleq_{LR} B$. □

Theorem (Kjos-Hanssen 2005)

The following are equivalent for reals A, B:

- $A \leq_{LR} B$
- *every $\Sigma^0_1(A)$ class of measure less than 1 is contained in some $\Sigma^0_1(B)$ class of measure less than 1*
- *some member of a universal Martin-Löf test relative to A is contained in a $\Sigma^0_1(B)$ class of measure less than 1.*

Ways $\leq_{LR}$ is like $\leq_T$:

- $\leq_{LR}$ is Σ^0_3 as a binary relation on reals.
- Barmpalias, Lewis, and Soskova have adapted techniques designed for c.e. Turing degrees to LR, such as Sacks coding and Sacks restraints.
- LR degrees are countable.
- $A \oplus B$ is an upper bound for A and B in the LR degrees.

Ways $\leq_{LR}$ is unlike $\leq_T$:

- $A \oplus B$ is not (in general) a least upper bound for A and B in the LR degrees. It is not known whether least upper bounds always exist.
- Some LR lower cones are uncountable. An example is the lower cone below $0'$

Open questions:

1. What reals have uncountably many predecessors?
2. Are there minimal degrees?
3. Do least upper bounds exist?
4. When l.u.b.s do exist, how do they compare with the Turing join?

Theorem (Diamondstone)

Given two low sets A, B, there is a low c.e. set C such that $A, B \leq_{LR} C$.

Contrast with Sacks' splitting theorem: there are two low sets whose Turing join is $0'$.

Consider the problem, given a Δ^0_2 set X, of producing a Δ^0_2 set Y, with $Y \geq_{LR} X$, possibly with some additional properties.

The obvious strategy:

- Start with $\Sigma^0_1(X)$ class U^X, member of universal ML-test relative to X.
- A Δ^0_2 approximation to X naturally gives an approximation to U^X.
- To make $Y \geq_{LR} X$, build $\Sigma^0_1(Y)$ class V^Y with $\mu(V^Y) < 1$ covering U^X.

The obvious strategy, continued:

- Start with $\Sigma^0_1(X)$ class U^X, member of universal ML-test relative to X.
- A Δ^0_2 approximation to X naturally gives an approximation to U^X.
- To make $Y \geq_{LR} X$, build $\Sigma^0_1(Y)$ class V^Y with $\mu(V^Y) < 1$ covering U^X.
- When N_σ enters U^X, we immediately put N_σ in V^Y.
- When N_σ leaves U^X, we have a hard choice:
  - We can do nothing. This preserves Y, which helps make Y weak. However, it keeps superfluous measure in V^Y. If done too often, we have $\mu(V^Y) = 1$.
  - We can change Y in order to remove N_σ from V^Y. This keeps superfluous measure out of V^Y, but threatens to make Y strong. If done too often, we have $Y \geq_T X$.

It is more natural to replace the measure μ by the weight w:

- The weight of a set of strings D is given by

$$w(D) = \sum_{\sigma \in D} 2^{-|\sigma|}$$

- If U is the open set generated by D, we have $w(D) \geq \mu(U)$, with equality iff D is prefix-free.
- We can redefine an ML-test (oracle ML-test) as a uniformly c.e. (in the oracle) sequence (U_n) where U_n is a set of strings with $w(U_n) \leq 2^{-n}$.
- We will call such a test universal if for any other test (U'_n), there is some $k \in \omega$ such that for all n, $U'_{n+k} \subseteq U_n$ (as a set of strings).

Theorem (Diamondstone)

Given two low sets A, B, there is a low c.e. set C such that $A, B \leq_{LR} C$.

- The idea is to enumerate a set C, and simultaneously build a c.e. in C set of strings V^C with $w(V^C) < 1$.
- We start with members of the universal relativized ML-test, U^A and U^B.
- Our goal is to make C low, and simultaneously $U^A \cup U^B \subseteq V^C$.
- The proof directly constructs C and V^C.

The strategy constructing C will meet the following requirements:

$$N_e : \quad (\exists^\infty s)\Phi_{e,s}^{C_s}(e)\downarrow \quad \Longrightarrow \quad \Phi_e^C(e)\downarrow$$

$$P_{2e} : \quad w(U^A \setminus V^C) \leq 2^{-e-4}$$

$$P_{2e+1} : \quad w(U^B \setminus V^C) \leq 2^{-e-4}$$

- The strategy for the negative requirements is the usual preservation method.
- The strategy for the positive requirement P_{2e} is to use low guessing to find a finite set D with weight $w(D) = 2^{-e-4}$ that the A' approximation thinks will remain in U^A, disjoint from sets held by higher priority positive requirements for A (if such a set exists).
- When D is found, enumerate it into V^C. If the D guess turns out to be wrong, change C to remove D, unless that violates a higher priority restraint.
- Lowness ensures this guess is eventually correct, so positive requirements become satisfied.

The tricky part is showing that the waste is small enough that $\mu(V^C) < 1$.

- Incorrect guesses add weight to V^C, when the low guessing method tells us some large set of strings will remain in $U^A \cup U^B$.
- Then restraint for a negative requirement holds this wasted weight permanently in V^C, even when the corresponding set leaves $U^A \cup U^B$.
- However, it is possible to ensure that the total wasted weight is small.

Thank you

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