Two equivalent definitions of random closed sets

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Classical probability theory studies random closed sets. Barmpalias, Brodhead, Cenzer, Dashti, and Weber introduced effective randomness for closed sets in [BBCDW] as the sets of paths through random extendible trees. Bjørn Kjos-Hanssen used a related definition obtained by effectivising a definition used in probability theory.

Closed sets are natural objects, and the study of random closed sets has been fruitful in probability theory. The same should be true in the theory of effective randomness. It is unfortunate having two different definitions. However, if the two definitions coincide, it provides evidence that they are the "right" definition.

Barmpalias et al.'s definition

Barmpalias et al. identified nonempty closed subsets of 2^ω with extendible trees.

- For any tree $T \subseteq 2^{<\omega}$, the set $[T]$ of paths through T is a closed set.
- For any closed set C, the tree $T_C = \{\sigma : C \cap N_\sigma \neq \emptyset\}$ is an extendible tree.
- This gives a natural one-to-one correspondance between closed sets and extendible trees.

For Barmpalias et al., a random closed set is a random extendible tree. To determine which trees were random, they used well-established definitions for random infinite strings - in this case, strings in the alphabet $\{0, 1, 2\}$, referred to as *ternary reals*.

Below each node in an extendible tree, there is either a left child (0), a right child (1), or 2 children. Which one occurs can be coded by 0, 1, or 2. Thus, a ternary real codes an extendible tree.

Definition

A *random BBCDW tree* is an extendible subtree of $2^{<\omega}$ coded by a Martin-Löf random ternary real. A *random (BBCDW) closed set* is the set of paths through some random BBCDW tree.

Below each node of a BBCDW tree, the possibilities of having only a left child, only a right child, and both children are equally likely.

Galton-Watson processes

In the 19th century, Francis Galton was interested in the question, "What is the probability of a surname becoming extinct?" Galton proposed a mathematical model for this situation, and asked for solutions. The first solution Galton found acceptable was proposed by H. W. Watson, and the model Galton created and Watson refined became known as the Galton-Watson process.

In a Galton-Watson process, a population comes in discrete generations, like the levels of a tree, beginning with a single individual. The numbers of children of each individual are independent identically distributed random variables on ω.

Theorem

In a Galton-Watson process, if the expected number of children is at most 1, the population will go extinct with probability 1 (except in the trivial case). If the expected number of children is greater than 1, there is a positive probability that the population will never go extinct.

Using the Galton-Watson process, we build a subtree of $2^{<\omega}$:

- Include the root at level 0.
- At each level n, go through the nodes to determine which children survive.
- For each child, flip a coin with probability p of heads. If the coin comes up heads, the child survives.
- Repeat for all n.

The probability p that a child survives is called the "survival parameter". If $p > \frac{1}{2}$, then the tree has a positive probability of being infinite.

This process is a sequence of binary decisions. We code as a sequence of 0s and 1s.

Definition

A *random GW tree with survival parameter p* is a subtree of $2^{<\omega}$ coded by a random binary real, relative to the weighted coinflip measure μ_p. A *random (GW) closed set (with parameter p)* is the set of paths through some random GW tree.

A random (GW) closed set may be empty, as opposed to random (BBCDW) closed sets, which are always nonempty. Classically, the definitions are the same: when $p = \frac{2}{3}$, both produce equivalent probability measures on the space of nonempty closed subsets of 2^ω.

However, there is no obvious reason that this should be true effectively. The trees produced by the two processes are quite different; one is extendible, while the other is filled with lots of "dead wood".

One is motivated to ask: do the two definitions coincide? Is one definition stronger than the other?

Two equivalent definitions

We answer this question:

Theorem

(Diamondstone, Kjos-Hanssen) The random closed sets studied by Barmpalias, Brodhead, Cenzer, Dashti, and Weber coincide with the nonempty random closed sets under the distribution studied by Kjos-Hanssen.

We look at the trees. In terms of trees, this theorem says that every random (BBCDW) tree is the extendible part of a random (GW) tree, and vice versa.

- To show that every random BBCDW tree is the extendible part of a random GW tree, we actually build the GW tree in question.
- For the other direction, we prove the contrapositive: given a nonrandom extendible tree T, and any tree T' such that the extendible part is T, then T' must not be random.
- We convert the test that T failed into a new test that T' will fail.

Turning BBCDW trees into GW trees

Our first goal is showing that every random BBCDW tree is the extendible part of a GW tree.

- BBCDW trees are extendible; GW trees are filled with dead wood.
- In order to turn the BBCDW tree into a GW tree, we must add dead wood.
- In order that the GW tree should be random, we will need to add dead wood which is random, relative to the BBCDW code.

Probability calclulations

There are some straightforward probability calculations we will make use of:

- Given that a node σ is part of the deadwood of a GW tree, we can compute the probabilities $P(\emptyset)$, $P(0)$, $P(1)$, and $P(2)$ that neither, only the left, only the right, and both children (respectively) are also in the tree. This defines a probability measure on 4, which in turn defines a measure μ_{dead} on 4^{ω^2}.
- Given that a node σ is in the live part of a GW tree, but σi is not, we can compute the probability q that σi is on the tree. This defines a probability measure on 2, which in turn defines a measure μ_{join} on 2^{ω}.

We define an effectively continuous map $\psi : 3^\omega \times 4^{\omega^2} \times 2^\omega \to 2^\omega$. To compute $\psi(f,g,h)$, where $f \in 3^\omega$, $g \in 4^{\omega^2}$, and $h \in 2^\omega$.

1. Build the BBCDW tree T coded by f.
2. Thinking of g as a sequence (g_i) of elements of 4^ω, build a sequence of trees (T_i), where each T_i coded by g_i.
3. For each i, let σ_i be the ith string not in T, but whose immediate predecessor is in T. If $h(i) = 1$, attach T_i to T below the σ_i. In other words, let

$$\hat{T} = T \cup \bigcup_{h(i)=1} \sigma_i T_i,$$

where $\sigma T = \{\sigma\tau : \tau \in T\}$.

4. Let $\psi(f,g,h)$ be the binary real which codes $\hat{T}$ (as a GW tree).

Theorem

Every random BBCDW tree is the extendible part of a random GW tree.

Proof.

- Let T be a BBCDW tree, coded by f.
- Let g be μ_{dead}-random relative to f. Let h be μ_{join}-random relative to $f \oplus g$.
- By van Lambalgen's theorem, (f, g, h) is $\lambda \times \mu_{\text{dead}} \times \mu_{\text{join}}$-random.
- Since ψ is effectively continuous and (almost) measure-preserving, $\psi(f, g, h)$ is μ_p-random.
- $\psi(f, g, h)$ codes a random GW tree with extendible part T.

□

Turning BBCDW tests into GW tests

For the other direction: the extendible part of every random GW tree is a random BBCDW tree.

Equivalently, we want to show that if an extendible BBCDW tree is non-random, then any tree formed by adding dead wood is also non-random.

We will convert the test that the BBCDW tree fails into a test that any GW tree formed by adding dead wood also fails.

What we would like to do, and what goes wrong

Suppose we have a test $U = \{U_n\}$ that our BBCDW tree fails.

- We would like to define a new test $V = \{V_n\}$:

$$V_n = \{T : \text{the extendible part of } T \text{ fails } U_n\}.$$

- Unfortunately, it is impossible to tell which parts of T are extendible and which are not in an effective way. (In fact, this V_n is not even open.)
- We must approximate to get the desired V.

Given a GW tree T, it is impossible to tell which parts are alive and which are dead. We must guess.

- We are mainly concerned with trees that appear to be (GW) random.
- We *can* make good guesses, given the assumption that T is random.
- Suppose we want to know T_{ext} through level l:
  1. Choose $\epsilon > 0$.
  2. Look for some large level L so that each node on T at level l either has no successors at level L, or enough successors that the probability of them all dying off is less than ϵ.
  3. Use L to approximate T_{ext} through level l.
  4. If the level L is not found, then there must be some node on T at level l such that there are always some successors, but never enough to push the probability of them all dying below ϵ: this is a measure 0 event, which we can test for.

Suppose we have a Martin-Löf λ-test $\{U_n\}$ on 3^ω.

- We think of this as being a test on extendible trees.
- We want to produce a μ_p test $\{V_n\}$ on 2^ω, which we will also think of as a test on trees.
- Ideally, T will fail $\{V_n\}$ if T_{ext} fails $\{U_n\}$.
- We define V_n to be the set of codes for trees T such that for some level l, the search described on the previous slide with $\epsilon = 2^{-n}$ terminates, and the approximation to T_{ext} through level l forces it to be in U_n.
- Then $\mu_p(V_n) \leq 2^{1-n}$, making $\{V_n\}$ a Martin-Löf μ_p-test.

Theorem

The extendible part of every infinite random GW tree is a random BBCDW tree.

Proof.

- Suppose not, and we have an infinite random GW tree T and a test U that the code for T_{ext} fails.
- Let V be the test approximating U.
- Since T is random, its code does not fail V.
- Since we know T_{ext} fails U, the approximations to T_{ext} must often be wrong.
- This is a measure 0 event we can test for. Contradiction.
- Therefore, the extendible part of T is a random BBCDW tree.

□

Thank you

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