ASYMPTOTIC DENSITY, COMPUTABLE TRACEABILITY, AND 1-RANDOMNESS

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Abstract. Let r be a real number in the unit interval $[0,1]$. A set $A \subseteq \omega$ is said to be *coarsely computable at density* r if there is a computable function f such that $\{n \mid f(n) = A(n)\}$ has lower density at least r. Our main results are that A is coarsely computable at density $1/2$ if A is either computably traceable or truth-table reducible to a 1-random set. In the other direction, we show that if a degree $\mathbf{a}$ is either hyperimmune or PA, then there is an $\mathbf{a}$-computable set which is not coarsely computable at any positive density.

1. Introduction

In recent years, a number of investigators have considered algorithms which frequently yield correct answers but may diverge or yield wrong answers on some inputs. Here "frequently" is often measured using (asymptotic) density or lower density, so we review the definitions of these.

For $A \subseteq \omega$, and $n > 0$, define

$$\rho_n(A) = \frac{|A \cap \{0, 1, \dots, n-1\}|}{n}.$$

The *upper density* of A, denoted $\overline{\rho}(A)$, is defined as $\limsup_n \rho_n(A)$ and the *lower density* of A, denoted $\underline{\rho}(A)$, is defined as $\liminf_n \rho_n(A)$. The *density* of A, denoted $\rho(A)$, is defined as $\lim_n \rho_n(A)$, provided that this limit exists. By the strong law of large numbers, almost every set (in the usual coin-toss measure on 2^ω) has density $1/2$. On the other hand, the sets A with $\underline{\rho}(A) = 0$ and $\overline{\rho}(A) = 1$ (and so $\rho(A)$ undefined) are comeager in the usual topology on 2^ω.

One major notion of frequent computability is generic computability. This has been applied to analyze the complexity in the generic case of decision problems in group theory (see, for example, [8]). A set $A \subseteq \omega$ is *generically computable* if there is a

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partial computable function ψ such that $\psi(n) = A(n)$ for *all* n in the domain of ψ, and this domain has asymptotic density 1. Generic computability for subsets of ω is studied in [6], and connections between asymptotic density and computability theory are studied in [3].

Suppose now that we wish to consider frequently correct algorithms which always yield an output. Then we must allow the possibility of some incorrect answers. A set A is *coarsely computable* if there is a (total) computable function f such that $\{x \mid A(x) = f(x)\}$ has density 1. Coarse computability and generic computability are independent in the sense that neither implies the other [6, Theorems 2.15 and 2.26].

Weakenings of these notions have also been considered, where sets of density 1 are replaced by sets whose lower density is at least a given number.

Definition 1.1. Let r be a real number in the interval $[0, 1]$ and let $A \subseteq \omega$.

(i) [3, Definition 5.9] A is *computable at density* r if there is a partial computable function φ such that $\varphi(n) = A(n)$ for all n in the domain of φ, and this domain has lower density at least r.

(ii) [4] A is *coarsely computable at density* r if there is a total computable function f such that $\{n \mid f(n) = A(n)\}$ has lower density at least r.

Note that we use lower density rather than upper density in these definitions since we wish our algorithms to function well from some point on, rather than just infinitely often. Also note that a set A is generically computable if and only if it is computable at density 1, and A is coarsely computable if any only if it is coarsely computable at density 1.

These definitions suggest measuring the complexity of a set A by considering $\{r \mid A \text{ is computable at density } r\}$, or the analogous set for coarse computability at density r. As these sets are closed downward in the unit interval, we instead just consider their sups.

Definition 1.2. Suppose $A \subseteq \omega$.

(i) [3, Definition 6.9] The *asymptotic computability bound* of A is

$$\alpha(A) := \sup\{r \mid A \text{ is computable at density } r\}.$$

(ii) [4] The *coarse computability bound* of A is

$$\gamma(A) := \sup\{r \mid A \text{ is coarsely computable at density } r\}.$$

As an example, note that if A is a 1-random set, then $\alpha(A) = 0$ and $\gamma(A) = 1/2$. In fact, to get that $\alpha(A) = 0$, it suffices to assume that A is weakly 1-random, and to get that $\gamma(A) = 1/2$ it suffices to assume that A is Schnorr random.

Note that if A is generically computable, then $\alpha(A) = 1$, and if A is coarsely computable, then $\gamma(A) = 1$. The converse of each statement fails. (This is proved for α in [3, Observation 5.10], and the same argument works for γ, since $\mathcal{R}(A)$, as defined there, is coarsely computable only when $A \leq_T 0'$ by [6, Theorem 2.19].)

It is shown in [6, Theorem 2.20] that every nonzero Turing degree contains a set which is neither coarsely computable nor generically computable. This suggests associating numbers with degrees $\mathbf{a}$ which calibrate the extent to which all sets of degree at most $\mathbf{a}$

are approximable by frequently correct algorithms. This turns out to be interesting only for coarse computability, as it turns out that every nonzero Turing degree contains a set which fails to be generically computable in a very strong sense, as explained in the next paragraph.

Miasnikov and Rybalov [10] defined a set A to be *absolutely undecidable* if every partial computable function φ such that $\varphi(x) = A(x)$ for all x in the domain of φ has a domain of density 0. (Note that this implies that $\alpha(A) = 0$, and it is easily seen that the converse fails.) Bienvenu, Day, and Hölzl [1] proved that every nonzero degree contains an absolutely undecidable set. Their proof uses an error-correcting code, the Walsh-Hadamard code.

However, it does turn out to be interesting to associate a number with each degree $\mathbf{a}$ which measures the extent to which all $\mathbf{a}$-computable functions are coarsely approximable, as we attempt to demonstrate in this paper.

Definition 1.3. The *coarse computability bound* of a **degree** $\mathbf{a}$ is given by:

$$\Gamma(\mathbf{a}) = \inf\{\gamma(A) \mid A \text{ is } \mathbf{a}\text{-computable}\}.$$

As mentioned, it was shown in [6, Theorem 2.20] that every nonzero degree contains a set which is not coarsely computable. It is natural to try to refine this by showing that $\Gamma(\mathbf{a})$ is "small" in some sense for every nonzero degree $\mathbf{a}$. The next result, due to Hirschfeldt, Jockusch, McNicholl and Schupp, is a step in that direction.

Proposition 1.4. *(*[4]*) If* $\mathbf{a}$ *is a nonzero degree, then* $\Gamma(\mathbf{a}) \leq 1/2$.

Proof. It suffices to show that for every noncomputable set A there is a set $B \equiv_T A$ such that $\gamma(B) \leq 1/2$. The idea is to code each bit of A by many bits of B so that an algorithm for B which is correct more than half the time yields an algorithm for A which is correct with only finitely many errors, by "majority vote."

For each n, let $I_n = \{k \in \omega \mid n! \leq k < (n+1)!\}$. For any set A, define

$$I(A) = \cup_{n \in A} I_n.$$

We claim that $I(A) \equiv_T A$ and $\gamma(I(A)) \leq 1/2$. The first statement is obvious. To see that $\gamma(I(A)) \leq 1/2$, assume for a contradiction that $I(A)$ is coarsely computable at some density greater than 1/2. Let f be a computable function such that $\{x \mid f(x) = I(A)(x)\}$ has lower density greater than 1/2. Then, for all sufficiently large n, we have $f(x) = I(A)(x)$ for strictly more than half of the elements of I_n. It follows that, for all sufficiently large n, n belongs to A if and only if $f(x) = 1$ for at least half of the numbers $x \in I_n$. Hence, A is computable, which is the desired contradiction. It follows that $\gamma(I(A)) \leq 1/2$. □

Let $I(A)$ be as defined in the above proof. Note that, for every A, $I(A)$ is coarsely computable at density 1/2, since $I(A)$ agrees with the set of even numbers on a set of lower density at least 1/2. It follows that $\gamma(I(A)) = 1/2$ for all noncomputable sets A. Hence, the above result cannot be improved without using a different construction. In the next few results, we give some improvements for certain classes of degrees.

Definition 1.5. (S. Kurtz [7]) A set A is *weakly* 1*-generic* if A meets every dense c.e. set of binary strings. (Here, if S is a set of binary strings, S is called *dense* if every string has an extension in S, and A *meets* S if (the characteristic function of) A extends some string in S.)

Proposition 1.6. *([4]) If A is weakly 1-generic, then $\gamma(A) = 0$.*

Proof. Let f be a computable function. We must show that $\{k \mid f(k) = A(k)\}$ has lower density 0. For each $n, j > 0$, define

$$S_{n,j} = \{\sigma \in 2^{<\omega} : |\sigma| \geq j \;\&\; \frac{\{k < |\sigma| \mid \sigma(k) = f(k)\}}{|\sigma|} < \frac{1}{n}\}.$$

Then each $S_{n,j}$ is computable and dense, so A meets each $S_{n,j}$. It follows that $\{k \mid f(k) = A(k)\}$ has lower density 0. □

Since Kurtz has shown [7, Corollary 2.10] that every hyperimmune set computes a weakly 1-generic set, we have the following corollary:

Corollary 1.7. *Every hyperimmune degree* $\mathbf{a}$ *satisfies* $\Gamma(\mathbf{a}) = 0$.

A degree $\mathbf{a}$ is called *PA* if every nonempty Π^0_1 class $P \subseteq 2^\omega$ has an $\mathbf{a}$-computable element. Many characterizations of the PA degrees can be found in [2, Section 2.21], for example.

Proposition 1.8. *If* $\mathbf{a}$ *is PA, then* $\Gamma(\mathbf{a}) = 0$.

Proof. Consider the Π^0_1 class

$$\{X \mid (\forall e)(\forall x \in I_e)[\varphi_e(x)\downarrow \;\rightarrow\; X(x) \neq \varphi_e(x)]\}$$

where $I_e = [e!, (e+1)!)$. It is easy to see that this class is nonempty, and for every X in the class, $\gamma(X) = 0$. Hence this class has an $\mathbf{a}$-computable element. □

Of course, it follows by well-known basis theorems that $\{\mathbf{a} \mid \Gamma(\mathbf{a}) = 0\}$ contains both hyperimmune-free and low degrees. This raises the question of whether this class contains all nonzero degrees. A positive answer would be a weak analogue of the Bienvenu-Day-Hölzl theorem [1] that every nonzero degree contains an essentially undecidable set. However, in this paper, we obtain a negative answer to this question in two different ways, and these are our main results. In fact, we prove that there is a nonzero degree $\mathbf{a}$ such that $\Gamma(\mathbf{a}) = 1/2$. The following definition, which is a uniform version of being hyperimmune-free, plays a key role in our first main result.

Definition 1.9. (Terwijn, Zambella [11]) The set A is *computably traceable* if there is a computable function p such that for every function $f \leq_T A$ there is a computable function g such that, for all n,

(i) $f(n) \in D_{g(n)}$
(ii) $|D_{g(n)}| \leq p(n)$

Here D_z is the finite set with canonical index z.

Note that p here must be independent of f. If the above holds, we say that A is computably traceable *via* p. As is shown in [11], if A is computably traceable, then A is computably traceable via every computable, nondecreasing, unbounded function h with $h(0) > 0$. Note that the standard construction of a hyperimmune-free degree with computable perfect trees, due to W. Miller and Martin [9], produces a set which is computably traceable via $\lambda n 2^n$. As pointed out in [11], this construction can easily be modified to show that there exist a continuum of computably traceable sets. A degree $\mathbf{a}$ is called *computably traceable* if there is a computably traceable set of degree $\mathbf{a}$, in which case every set of degree $\mathbf{a}$ is also computably traceable. The computably traceable sets have played an important role in the study of algorithmic randomness, as explained in [2, Chapter 12].

Our first main result is the following:

Theorem 1.10. *If the set A is computably traceable, then A is coarsely computable at density* $1/2$.

Corollary 1.11. (i) *If $\mathbf{a}$ is a nonzero computably traceable degree, then* $\Gamma(\mathbf{a}) = 1/2$.
(ii) *There is a degree $\mathbf{a}$ such that $\mathbf{a} \leq \mathbf{0}''$ and* $\Gamma(\mathbf{a}) = 1/2$.
(iii) *There exist continuum many degrees $\mathbf{a}$ such that* $\Gamma(\mathbf{a}) = 1/2$.

Our second main result is the following:

Theorem 1.12. *If the set X is* 1*-random and A is truth-table reducible to X, then A is coarsely computable at density* $1/2$.

Corollary 1.13. (i) *If $\mathbf{x}$ is a hyperimmune-free* 1*-random degree, then* $\Gamma(\mathbf{x}) = 1/2$.
(ii) *There is a DNC degree $\mathbf{x} \leq \mathbf{0}''$ such that* $\Gamma(\mathbf{x}) = 1/2$.

Proof. For (i), let X be a 1-random set of degree $\mathbf{x}$. By a result of D. A. Martin (see [2, Proposition 2.17.7]), if $A \leq_T X$ then $A \leq_{tt} X$, since $\mathbf{x}$ is hyperimmmune-free. It follows from the theorem that $\Gamma(\mathbf{x}) \geq 1/2$, and $\Gamma(\mathbf{x}) \leq 1/2$ by Proposition 1.4.

To prove (ii), let $P \subseteq 2^\omega$ be a non-empty Π^0_1 class such that every element of P is a 1-random set. Then P has an element $X \leq_T 0''$ of hyperimmune-free degree, by the hyperimmune-free basis theorem (see [2, Theorem 2.19.11]) and its proof. If $\mathbf{x}$ is the degree of X, then $\Gamma(\mathbf{x}) = 1/2$ by part (i), and $\mathbf{x}$ is DNC by Kučera's theorem that every 1-random set computes a DNC function (see [2, Theorem 8.8.1]). □

To summarize, we know that $\Gamma(\mathbf{0}) = 1$, $\Gamma(\mathbf{a}) \leq 1/2$ for all $\mathbf{a} > \mathbf{0}$, $\Gamma(\mathbf{a}) = 0$ for all degrees which are hyperimmune or PA, and $\Gamma(\mathbf{a}) = 1/2$ for every degree $\mathbf{a}$ which is either nonzero and computably traceable or hyperimmune-free and 1-random. We do not know whether Γ takes values other than 0, 1/2, and 1.

2. Proof of our first main result

In this section we prove Theorem 1.10, except for a combinatorial lemma which is proved in the next section. We start by partitioning the natural numbers into consecutive intervals $J_1, J_2, \ldots$, where $|J_n| = n$ for all n. If A is computably traceable, we can effectively find a set T_n of n strings of length n such that some string in T_n describes $A \upharpoonright J_n$. Then using our combinatorial lemma we can effectively find a string β_n which approximates *all* strings in T_n with only slightly more than $n/2$

errors. Then concatenating these strings β_n in order yields a computable set B such that $\underline{\rho}(\{k \mid A(k) = B(k)\}) \geq 1/2$ so that A is coarsely computable at density 1/2.

We now give the details of the argument. In the Hamming space 2^n, we define the (normalized) distance between two strings σ and τ of length n to be:

$$d(\sigma, \tau) = \frac{|\{k < n \mid \sigma(k) \neq \tau(k)\}|}{n}.$$

If $\sigma \in 2^n$ and T is a nonempty subset of 2^n, we define the distance from σ to T to be

$$\hat{d}(\sigma, T) = \max\{d(\sigma, \tau) \mid \tau \in T\}.$$

Thus the distance between a string and a set of strings of the same length is the *greatest* distance between the string and any string in the set.

Lemma 2.1. *Let ϵ be a positive real number. Then if n is sufficiently large and T is a set of n strings of length n, there exists $\sigma \in 2^n$ such that $\hat{d}(\sigma, T) \leq 1/2 + \epsilon$.*

Intuitively, given any tolerance $\epsilon > 0$, if n is sufficiently large, we can "approximate" any n given strings of length n by a single string of length n which is at distance at most $1/2 + \epsilon$ from each of the given strings.

The lemma follows easily from a convergence bound (Chernoff's Inequality) for the weak law of large numbers. We will prove it in Section 3. In fact, we will show by probabilistic reasoning that for any $\epsilon > 0$ and any sufficiently large n, for any set T of n strings of length n, "most" strings σ of length n satisfy the conclusion of the lemma, because the probability of not satisfying it is so small. Of course, such probabilistic arguments are frequently used in combinatorics. This lemma will be proved in the next section.

For the rest of this section, we focus on using the above lemma to prove Theorem 1.10, which asserts that every computably traceable set is coarsely computable at density 1/2.

Proof of Theorem 1.10. Let A be a computably traceable set. We identify A with the infinite binary sequence $A(0)A(1)...$, where $A(i) = 1$ if and only if $i \in A$. Let this sequence be decomposed as $\alpha_1 {}^\frown \alpha_2 {}^\frown \dots$, where α_i is a binary string of length i. For example, α_3 is the string $A(3)A(4)A(5)$. Since A is computably traceable, there are uniformly and canonically computable finite sets $T_1, T_2, \dots$ such that $\alpha_n \in T_n$ and $|T_n| \leq n$ for all $n > 0$. Here we may assume without loss of generality that each T_n is a set of n strings of length n.

We now wish to define a computable set B such that $\{n \mid A(n) = B(n)\}$ has lower density at least 1/2. We define (using the same identifications as for A) $B = \beta_1 {}^\frown \beta_2 {}^\frown \dots$, where β_n is a string of length n which is as close to T_n as possible, that is $\hat{d}(\beta_n, T_n) \leq \hat{d}(\beta, T_n)$ for all $\beta \in 2^n$. It is clear that such a closest string exists and can be chosen effectively, so we may make B computable by always picking the least candidate for β_n. Thus we are making B close to A by making each β_n as close as possible to T_n, where $\alpha_n \in T_n$.

Let $C = \{k \mid B(k) = A(k)\}$. We claim that $\underline{\rho}(C) \geq 1/2$, so that A is computable at density 1/2. Let t_n be the nth triangular number $n(n+1)/2$, so that t_n is the length of $\beta_1 {}^\frown \beta_2 {}^\frown \dots {}^\frown \beta_n$. If F is a nonempty finite set, define the *density of C on F*, denoted $\rho(C|F)$, to be $\frac{|C\cap F|}{|F|}$. We first consider the density of C on the intervals J_n, where $J_1 = \{0\}$ and $J_n = [t_{n-1}, t_n)$ for $n > 0$, so $|J_n| = n$ for all n.

Lemma 2.2. *$\liminf_n \rho(C|J_n) \geq 1/2$.*

Proof. To prove the lemma, let $\epsilon > 0$ be given. We must show that $\rho(C|J_n) \geq 1/2 - \epsilon$ for all sufficiently large n. By definition,

$$\rho(C|J_n) = \frac{|\{k \in J_n \mid A(k) = B(k)\}|}{n} = \frac{|\{k < n \mid \alpha_n(k) = \beta_n(k)\}|}{n} = 1 - d(\alpha_n, \beta_n).$$

Also, for all sufficiently large n, $d(\beta_n, \alpha_n) \leq \hat{d}(\beta_n, T_n) \leq 1/2 + \epsilon$ by Lemma 2.1. Hence, as needed, it follows that $\rho(C|J_n) \geq 1/2 - \epsilon$ for all sufficiently large n. □

We now consider the lower density of C on sets of the form $\cup_{i\leq n} J_i = [0, t_n)$.

Lemma 2.3. *$\liminf_n \rho_{t_n}(C) \geq 1/2$.*

Proof. Let $\epsilon > 0$ be given. We must show that $\rho_{t_n}(C) \geq 1/2 - \epsilon$ for all sufficiently large n. By the previous lemma, we have $\rho(C|J_n) \geq 1/2 - \epsilon/2$ for all sufficiently large n. Hence, there is a finite set F such that $\rho(C \cup F|J_n) \geq 1/2 - \epsilon/2$ for *all* n. Note that $\rho_{t_n}(C \cup F)$ is a weighted average of the numbers $\rho(C \cup F|J_i)$ for $i \leq n$. Since all the latter numbers are $\geq 1 - \epsilon/2$, it follows that $\rho_{t_n}(C \cup F) \geq 1 - \epsilon/2$ for all n. Since F is finite, $\rho_{t_n}(F) \leq \epsilon/2$ for sufficiently large n. Hence we have $\rho_{t_n}(C) \geq 1/2 - \epsilon$ for all sufficiently large n, which establishes the lemma. □

We now must consider values of $\rho_k(C)$, when k is not a triangular number. These values are easily reduced to the previous case because the triangular numbers grow slowly, in the sense that $\lim_n \frac{t_{n+1}}{t_n} = 1$. Specifically, suppose that $t_n < k \leq t_{n+1}$. Then

$$\rho_k(C) = \frac{|C \cap \{0, 1, \ldots, k-1\}|}{k} \geq \frac{t_n \cdot \rho_{t_n}(C)}{t_{n+1}}.$$

As k tends to infinity, n also tends to infinity, and $\frac{t_n}{t_{n+1}}$ tends to 1, so

$$\underline{\rho}(C) = \liminf_k \rho_k(C) \geq \liminf_n \rho_{t_n}(C) \geq 1/2$$

as needed to complete the proof of the theorem. □

3. Proof of Lemma 2.1

We use a probabilistic argument to prove our combinatorial lemma, Lemma 2.1. Suppose a fair coin is thrown n times. Let p_n be the probability that heads are obtained on at most 49% of the throws. Then, by the weak law of large numbers, $\lim_n p_n = 0$. Of course, the same holds if we replace 49% by any fixed real number less than 1/2. The key to proving Lemma 2.1 is Chernoff's inequality, which shows that p_n goes to 0 exponentially fast. We write $P(A)$ for the probability of the event A when the intended probability space is clear from context.

Theorem 3.1. *(Chernoff's Inequality). (See* [10, Theorem 4.2]*.) Let the random variable S be binomially distributed with parameters n and p, so we can think of S as the number of heads obtained in n independent tosses of a possibly biased coin, where p*

is the probability of heads on each individual toss. Let μ *be the expected value of* S*, so* $\mu = np$*. Suppose* $0 \leq \delta \leq 1$*. Then*

$$P(S < (1-\delta)\mu) \ < \ e^{-\mu\delta^2/2}.$$

Proof of Lemma 2.1. Let $\epsilon > 0$ be given and let T be a set of n binary strings of length n. To prove Lemma 2.1 we wish to show that if n is sufficiently large (depending only on ϵ), there is a string $\sigma \in 2^n$ with $\hat{d}(\sigma, T) < 1/2 + \epsilon$, i.e., $d(\sigma, \tau) < 1/2 + \epsilon$ for all $\tau \in T$. Let 0^n be the string of length n consisting of all 0's. Define

$$b_{n,\epsilon} = 2^{-n} |\{\sigma \in 2^n \mid d(\sigma, 0^n) < 1/2 - \epsilon\}|.$$

Thus $b_{n,\epsilon}$ represents the probability that a string $\sigma \in 2^n$ chosen uniformly at random has fewer than $n(1/2 - \epsilon)$ 1's. By the homogeneity of Hamming space, $b_{n,\epsilon}$ would have the same value if 0^n were replaced in its definition by any fixed string $\tau \in 2^n$. Thus, for each string $\tau \in 2^n$

$$P(d(\sigma, \tau) < 1/2 - \epsilon) = b_{n,\epsilon} \tag{1}$$

for $\sigma \in 2^n$ chosen uniformly at random.

Now define the random variable S_n as the number of 1's in a string $\sigma \in 2^n$ chosen uniformly at random. Thus $S_n = nd(\sigma, 0^n)$, where σ is chosen uniformly at random. We can think of σ as determined by n tosses of a fair coin, so S_n is a binomially distributed random variable with parameters n and $1/2$ and $\mu = n/2$. Then by Chernoff's inequality applied to S_n with $\delta = 2\epsilon$,

$$P(S_n < n(1/2 - \epsilon)) = P(S_n < (1 - 2\epsilon)\frac{n}{2}) \ < \ e^{-(n/2)(2\epsilon)^2/2}.$$

Since $P(S_n < (1 - 2\epsilon)\frac{n}{2}) = b_{n,\epsilon}$ by definition of $b_{n,\epsilon}$, we have

$$b_{n,\epsilon} < e^{-n\epsilon^2} \tag{2}$$

Fix $\tau \in 2^n$. Let $\overline{\tau}$ be the string of length n which is complementary to τ, so $\overline{\tau}(i) = 1$ if and only if $\tau(i) = 0$ for $i < n$. Note that, for every $\sigma \in 2^n$, $d(\sigma, \tau) = 1 - d(\sigma, \overline{\tau})$. Hence, if $\sigma \in 2^n$ is chosen uniformly at random,

$$P(d(\sigma, \tau) > 1/2 + \epsilon) = P(d(\sigma, \overline{\tau}) < 1/2 - \epsilon) = b_{n,\epsilon} \tag{3}$$

where the final equality uses Equation (1).

Suppose again that σ is chosen uniformly at random from 2^n. For each fixed $\tau \in T$, by Equations (2) and (3), the probability that $d(\sigma, \tau) > 1/2 + \epsilon$ is at most $e^{-n\epsilon^2}$. Since $|T| = n$ and the probability of a finite union of events is at most the sum of their probabilities, the probability that there exists $\tau \in T$ with $d(\sigma, \tau) > 1/2 + \epsilon$ is at most $ne^{-n\epsilon^2}$. It follows that the probability that $\hat{d}(\sigma, T_n) \leq 1/2 + \epsilon$ is at least $1 - ne^{-n\epsilon^2}$. Since the latter approaches 1 as n as approaches infinity, it is positive for all sufficiently large n. Hence, for all sufficiently large n, there exists $\sigma \in 2^n$ such that $\hat{d}(\sigma, T_n) \leq 1/2 + \epsilon$, as needed to prove Lemma 2.1. ☐

Remark. Instead of using Chernoff's Inequality, we could instead use Chebyshev's Inequality, which is better known but not as powerful in our context. It asserts that if X is a random variable with mean μ and finite variance σ, then

$$P(|X - \mu| \geq t) \ \leq \ \sigma^2/t^2.$$

From this we calculate that, if $\sigma \in 2^n$ is chosen randomly, then

$$P(d(\sigma, 0^n) < 1/2 - \epsilon) \;\; < \;\; 1/(4n\epsilon^2)$$

(see [5, pages 101-102]). If we then require of the trace $\{T_n\}$ that $|T_n| = o(n)$, i.e. $\lim_n \frac{|T_n|}{n} = 0$, and weaken the statement of Lemma 2.1 accordingly, the proof still goes through.

4. Proof of Theorem 1.12

In this section we prove Theorem 1.12, which asserts that if A is a set which is truth-table reducible to some 1-random set, then A is coarsely computable at density 1/2. We use a characterization of 1-randomness due to Solovay (see [2], Theorem 6.2.8). Namely, a *Solovay test* is a sequence $\{S_n\}$ of uniformly Σ^0_1 subsets of 2^ω such that $\sum_n \mu(S_n)$ converges, where μ is Lebesgue measure. A set X *passes* this test if X belongs to S_n for only finitely many n. Then X is 1-random if and only if X passes every Solovay test.

Fix a truth-table functional Φ, i.e., Φ is a Turing functional, and Φ^X is total for every set $X \subseteq \omega$. Assume that $A = \Phi^Y$ for some 1-random set Y. Our goal is to give a Solovay test $\{S_n\}$ such that Φ^X is coarsely computable at density 1/2 for every set X which passes the test. Since Y is 1-random, it must pass the test $\{S_n\}$ and hence $\Phi^Y = A$ is coarsely computable at density 1/2. In fact, we give a computable set B (dependent only on Φ) such that the lower density of $\{k \mid \Phi^X(k) = B(k)\}$ is at least 1/2 for every set X which passes the test. As in the proof of Theorem 1.10, we obtain B as $\beta_1 {}^\frown \beta_2 {}^\frown \dots$ where β_n is a string of length n for each n. For each set X, let Φ^X be decomposed as $\alpha_1^X {}^\frown \alpha_2^X {}^\frown \dots$, where each α_n^X is a string of length n. Let $\epsilon_1 = \epsilon_2 = 1/2$ and $\epsilon_n = 1/\log n$ for $n \geq 3$. (These numbers are chosen to be sufficiently small that $\lim_n \epsilon_n = 0$ and yet sufficiently large that we can eventually use Chernoff's Inequality to show that our $\{S_n\}$ is a Solovay test.) We now choose β_n so as to maximize the probability β_n and α_n^X agree on at least $n(1/2 - \epsilon_n)$ arguments. In more detail, for each string β of length n, let $m(n, \beta)$ be the Lebesgue measure of the set of X such that α_n^X and β agree on at least $n(1/2 - \epsilon_n)$ arguments. Note that m is a computable function of n and β. Define β_n so that $m(n, \beta_n) \geq m(n, \beta)$ for all $\beta \in 2^n$. Then $B = \beta_1 {}^\frown \beta_2 {}^\frown \dots$ is a computable set.

Let S_n be the set of X such that α_n^X and β_n disagree on more than $n(1/2 + \epsilon_n)$ arguments. We will show that $\{S_n\}$ is a Solovay test, but we defer the proof of this for now.

Fix a set X which passes the test $\{S_n\}$, i.e., X belongs to S_n for only finitely many n. Let $A = \Phi^X$, and let $C = \{k \mid A(k) = B(k)\}$. We will show that C has lower density at least 1/2. The next lemma is a special case of this.

Lemma 4.1. $\liminf_n \rho_{t(n)}(C) \geq 1/2$.

Proof. If $\epsilon > 0$, we have $\rho(C|J_n) \geq 1/2 - \epsilon$ for all sufficiently large n, since $\rho(C|J_n) \geq 1/2 - \epsilon_n$ for all sufficiently large n, and $\lim_n \epsilon_n = 0$. The rest of the proof is identical to that of Lemma 2.3. ☐

It follows from this lemma that $\underline{\rho}(C) \geq 1/2$ by the same argument that the corresponding fact is proved in the last paragraph of the proof of Theorem 1.10.

Since every 1-random set passes every Solovay test, it remains only to show that $\{S_n\}$ is a Solovay test. Clearly each S_n is a clopen set, uniformly effectively in n. Thus it remains only to show that $\sum_n \mu(S_n)$ is convergent. Note that $\mu(S_n) = 1 - m(n, \beta_n)$.

As in Section 3, let $b_{n,\epsilon}$ denote the probability that a string σ chosen uniformly at random from the strings of length n has fewer than $n(1/2 - \epsilon)$ 1's. By Equation (2) in Section 3, for each $\tau \in 2^n$, $b_{n,\epsilon}$ is also the probability that a string σ chosen uniformly at random from 2^n satisfies $d(\sigma, \tau) > 1/2 + \epsilon$.

If our functional Φ were the identity functional, we would have $m(n, \sigma) = 1 - b_{n,\epsilon_n}$ for every string σ of length n, since the measure given by Φ would be the uniform measure. Hence, in this special case, we would have $\mu(S_n) = b_{n,\epsilon_n}$. The next lemma will imply that, for a general Φ, there is *some* string $\sigma \in 2^n$ with $m(n, \sigma) \geq 1 - b_{n,\epsilon_n}$ and hence $\mu(S_n) \leq b_{n,\epsilon_n}$.

Lemma 4.2. *Suppose we are given $n \in \omega$ and a positive real number ϵ. Further, suppose we are given real numbers p_σ for each $\sigma \in 2^n$ such that $\sum_{\sigma \in 2^n} p_\sigma = 1$. For each $\sigma \in 2^n$, define:*

$$q_\sigma = \sum \{p_\tau \mid d(\tau, \sigma) \leq 1/2 + \epsilon\}$$

where d is normalized Hamming distance. Then there exists $\beta \in 2^n$ such that $q_\beta \geq 1 - b_{n,\epsilon}$.

Proof. We calculate the average value v of q_σ over all $\sigma \in 2^n$. We have

$$v = 2^{-n} \sum \{q_\sigma \mid \sigma \in 2^n\}.$$

Note that each summand of the above sum is itself a sum of terms of the form p_τ. Further, each p_τ occurs in $2^n(1 - b_{n,\epsilon})$ summands of v, where $2^n(1 - b_{n,\epsilon})$ does not depend on τ so that

$$v = 2^{-n} 2^n (1 - b_{n,\epsilon}) \sum_{\tau \in 2^n} p_\tau = 1 - b_{n,\epsilon}$$

Clearly, there must exist some $\beta \in 2^n$ such that q_β is at least the average value $v = 1 - b_{n,\epsilon}$ of these quantities. □

We now apply the lemma with $\epsilon = \epsilon_n$ and $p_\sigma = \mu(\{X \mid \alpha_n^X = \sigma\})$ for each $\sigma \in 2^n$. Let β be the resulting string with $q_\beta \geq 1 - b_{n,\epsilon_n}$. For every string $\sigma \in 2^n$, we have $m(n, \sigma) = q_\sigma$, so $m(n, \beta_n) \geq m(n, \beta) = q_\beta \geq 1 - b_{n,\epsilon_n}$. It follows that $\mu(S_n) = 1 - m(n, \beta_n) \leq b_{n,\epsilon_n}$.

We have, by Equation (2) in Section 3 that

$$b_{n,\epsilon_n} < e^{-n\epsilon_n^2} = e^{-\frac{n}{(\log n)^2}}.$$

Since $\sum_n e^{-\frac{n}{(\log n)^2}}$ converges, it follows that $\sum_n b_{n,\epsilon_n}$ converges. Hence, by comparison, $\sum_n \mu(S_n)$ converges, and $\{S_n\}$ is a Solovay test, which completes the proof.

Remark. After the proof of Theorem 1.10 we remarked that we could have used Chebyshev's Inequality in place of Chernoff's Inequality by making a small adjustment. However, it does not seem possible to do this in the current result. The natural change to make would be to make $\{\epsilon_n\}$ approach 0 more slowly. However, the natural upper bound on $\mu(S_n)$ we get from Chebyshev's inequality is $1/(4n\epsilon_n^2)$, and it is impossible to choose $\{\epsilon_n\}$ so that it tends to 0 and $\sum_n(1/4n\epsilon_n^2)$ converges, by the divergence of the harmonic series.

5. Open Questions

Let C_1 be the set of degrees $\mathbf{a}$ such that either $\mathbf{a}$ is computably traceable or $\mathbf{a}$ is both 1-random and hyperimmune-free. Let C_2 be the set of degrees which are neither hyperimmune nor PA. By the results of this paper

$$C_1 \ \subseteq\ \{\mathbf{a} \mid \Gamma(\mathbf{a}) \geq 1/2\} \ \subseteq\ C_2.$$

Question 5.1. *Can either of the two inclusions above be replaced by equality?*

Note that $\{\mathbf{a} \mid \Gamma(\mathbf{a}) \geq 1/2)\}$ is closed downward, so that for this class to equal C_i, where $i \in \{1, 2\}$, it is necessary that C_i be closed downward. It is clear that C_2 is closed downward. Demuth proved (see [2, Theorem 8.6.1]) that every noncomputable set truth-table reducible to a 1-random set has 1-random Turing degree. From this, it easily follows that C_1 is also closed downward.

Question 5.2. *What is the range R of Γ?*

We know only that $\{0, 1/2, 1\} \subseteq R \subseteq [0, 1/2] \cup \{1\}$.

Question 5.3. *If $\Gamma(\mathbf{a}) = 1/2$, must every $\mathbf{a}$-computable set be coarsely computable at density $1/2$?*

Theorems 1.10 and 1.12 show that if $\mathbf{a}$ is computably traceable or 1-random and hyperimmune-free, then every $\mathbf{a}$-computable set is coarsely computable at density $1/2$, so these results do not suffice to answer this question.

References

1. L. Bienvenu, Adam R. Day, and Rupert Hölzl, *From bi-immunity to absolute undecidability*, to appear in J. Symbolic Logic
2. Rodney G. Downey and Denis R. Hirschfeldt, *Algorithmic randomness and complexity*, Theory and Applications of Computability, Springer, New York, 2010.
3. Rodney G. Downey, Carl G. Jockusch, Jr., and Paul E. Schupp, *Asymptotic density and computably enumerable sets*, Journal of Mathematical Logic, **13**, No. 2 (2013), 43 pages.
4. Denis R. Hirschfeldt, Carl G. Jockusch, Jr., Timothy McNicholl, and Paul E. Schupp, *Asymptotic density and the coarse computability bound*, in preparation.
5. Paul G. Hoel, Sidney C. Port, and Charles J. Stone, *Introduction to Probability Theory*, Houghton Mifflin Company, Boston, 1971.
6. Carl G. Jockusch, Jr. and Paul E. Schupp, *Generic computability, Turing degrees, and asymptotic density*, J. Lond. Math. Soc. (2) **85** (2012), no. 2, 472–490.
7. Stuart A. Kurtz, *Notions of weak genericity*, J. Symbolic Logic **48** (1983), 764-770.
8. Ilya E. Kapovich, Alexei G. Miasnikov, Paul E. Schupp, and Vladimir E. Shpilrain, *Generic case complexity, decision problems in group theory and random walks*, J. Algebra **264** (2003), 665–694.
9. Webb Miller and Donald A Martin, *The degrees of hyperimmune sets*, Z. Math. Logik Grundlagen Math. **14** (1968), 159–166.
10. Rajeev Motwani and Prabhakar Raghavan, *Randomized Algorithms*, Cambridge University Press, Cambridge, New York, 1995.

11. Sebastiaan A. Terwijn and Domenico Zambella, *Algorithmic randomness and lowness*, J. Symbolic Logic **66** (2001), 1199–1205

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